NON OSCILLATORY INTERPOLATION METHODS APPLIED TO KINETIC EQUATIONS FOR PLASMAS

Francesco Vecil

Project work directed by Prof. José Antonio Carrillo de la Plata

fvecil@mat.uab.es

Universitat Autònoma de Barcelona

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- PWENO Interpolation

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- Linear Advection and Tests

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Introduction

We shall simulate three Vlasov-based models.

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = \mathcal{Q}[f] \\ f_0(x,v) = f(t=0,x,v). \end{cases}$$

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Both in dividing the solution of the Vlasov part from the Boltzmann part and in diving the solution in either directions in Vlasov's equation, we shall use Strang's splitting techniques. In splitting Vlasov's equation, this will lead us to solving linear advection equation.

Vlasov-Boltzmann

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We shall use two different initial functions for the simulations.

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So, the model is

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Landau damping

In the second model we are going to analyze, the force field is computed beginning from Poisson's equation

$$\frac{\partial^2 \Phi}{\partial x^2} = \left(1 - \int_{\mathbb{R}} f(t,x,v)dv\right),$$

with periodicity conditions at the border.

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This model should couserve the total energy

$$\int_\Omega \int_{\mathbb{R}} \frac{v^2}{2} f(t,x,v)dvdx + \frac{1}{2}\int_\Omega \rho\Phi^{self}dx + \int_\Omega \rho\Phi^{ext}dx.$$

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$$f_0(x,p) = (1+\epsilon\cos(k_e x))G(p-\epsilon v_{th}\cos(k_e x)),$$

where

$$G(p) = \alpha \frac{1}{\sqrt{2\pi} v_{th}} e^{-\frac{p^2}{2v_{th}^2}} + \frac{1-\alpha}{z} e^{-\frac{\sqrt{1+p^2}-1}{k_B T_{hot}}}.$$

Vlasov-Maxwell

So, the model is

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{p}{\sqrt{1+p^2}} \frac{\partial f}{\partial x} - \left(\eta^{-1} E + A \frac{\partial A}{\partial x}\right) \frac{\partial f}{\partial p} = 0 \\ \Phi_{xx} = \eta^{-2} \left[n^{ext} - \rho \right], \qquad \frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial x^2} = -\eta^{-2} \rho A \\ \mathcal{E} = -\frac{\partial A}{\partial t}, \qquad \frac{\partial \mathcal{E}}{\partial x} = -\frac{\partial B}{\partial t} \\ -\frac{\partial B}{\partial x} = -\eta^{-2} A \int_{\mathbb{R}} f\, dp + \frac{\partial \mathcal{E}}{\partial t} \\ f_0(x,p) = f(t=0,x,p). \end{cases}$$

This model should conserve the total energy, given by the sum of

$$\begin{cases} WT(t) = \frac{1}{2} \int_{[0,1]} \rho A^2 dx + \frac{1}{2} \eta^2 \int_{[0,1]} \left[\mathcal{E}^2 + B^2 \right] dx \\ WL(t) = \int_{[0,1]} \int_{\mathbb{R}} \sqrt{1+p^2} f\, dp dx + \frac{1}{2} \eta^2 \int_{[0,1]} E^2 dx. \end{cases}$$

PWENO interpolations

- Basic idea of PWENO interpolation

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Basic idea

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The main idea is to make a convex combination of several Lagrange polynomials, each of them interpolating $f(x)$ at certain points.

$$p^W(x) = \sum_{r=0}^{nlp-1} \omega_r(x) p_r(x),$$

where $p_r(x)$ are the Lagrange polynomials interpolating at the stencils $\mathcal{S}_r$.

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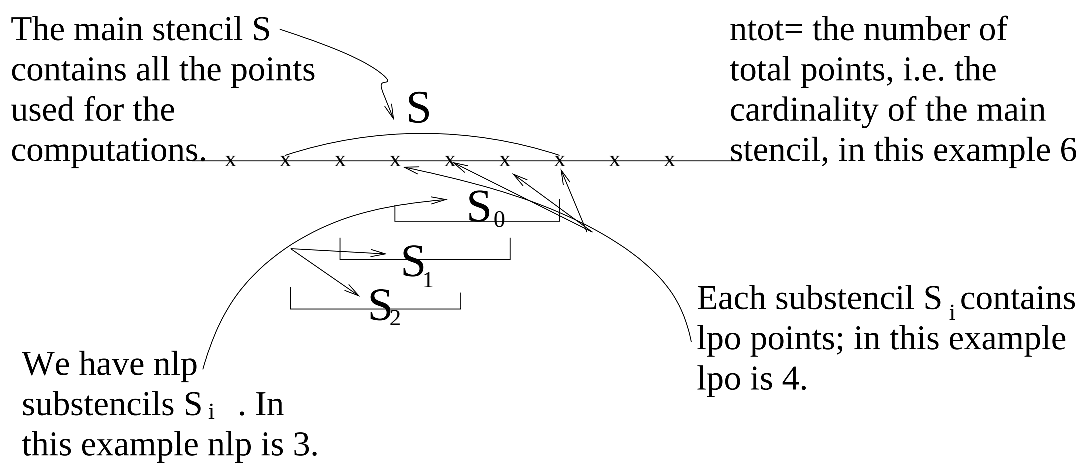
where $p_r(x)$ are the Lagrange polynomials interpolating at the stencils $\mathcal{S}_r$.
The interpolation is wanted to be non-oscillatory: where high gradients are produced, no spurious oscillations shall appear, and the total variation must be controlled.

Parameters


The main stencil S contains all the points used for the computations.
S
ntot= the number of total points, i.e. the cardinality of the main stencil, in this example 6
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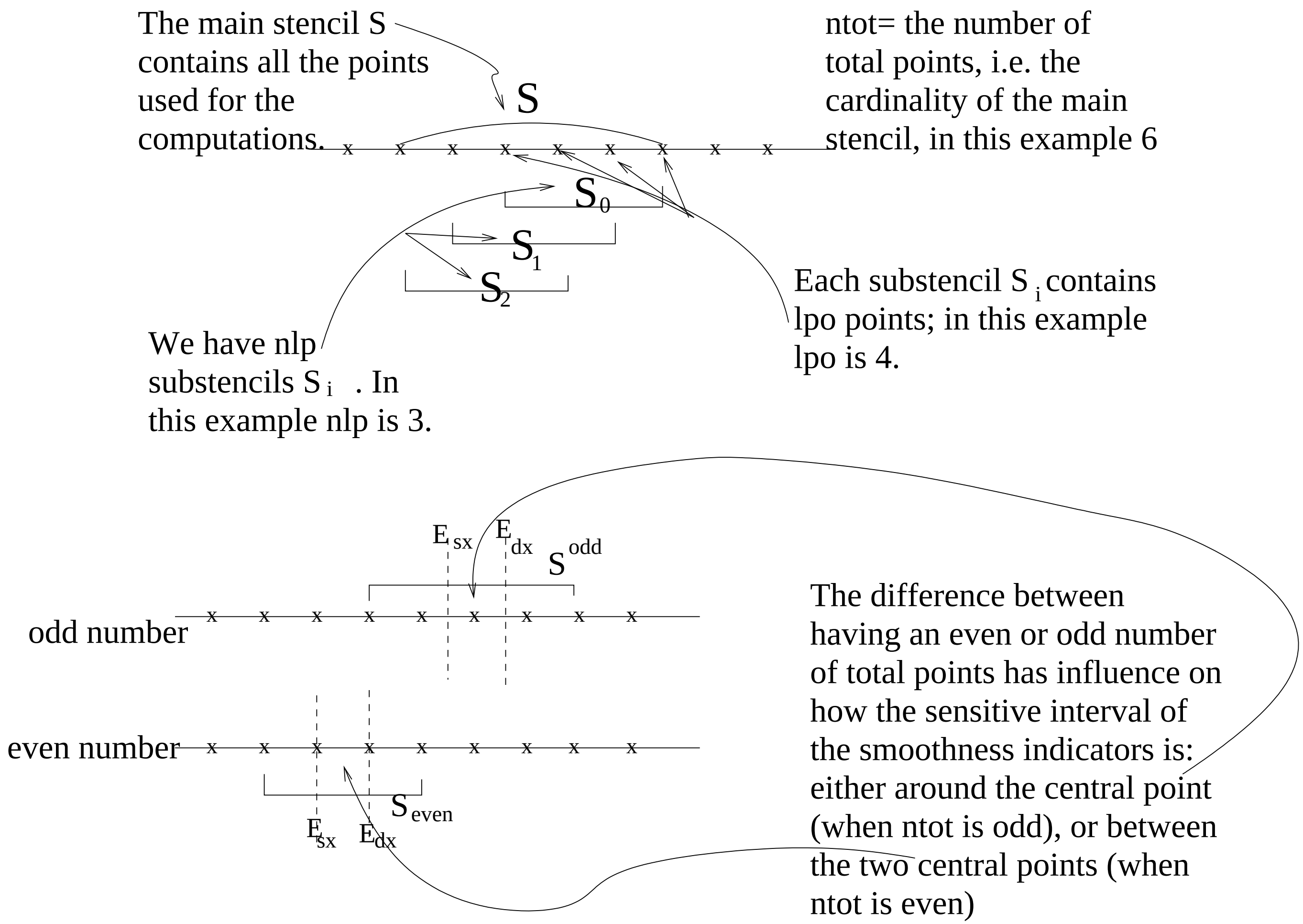
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E sx
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S odd
odd number
even number
S even
E sx
E dx
The difference between having an even or odd number of total points has influence on how the sensitive interval of the smoothness indicators is: either around the central point (when ntot is odd), or between the two central points (when ntot is even)

Smoothness indicators β_r

A measure of the regularity of Lagrange polynomials near the interpolation point is needed. This is obtained by a sort of weighted Sobolev norm.

$$\beta_r = \sum_{l=1}^{lpo-1} \Delta x^{2l-1} \left\| \frac{d^l}{dx^l} p_r \right\|^2_{L^2(\mathcal{E}_{sx}, \mathcal{E}_{dx})}.$$

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It is not the only possible choice, but this one is easy-implementable and the most used.

Weights $d_r(x)$

Let $p(x)$ the Lagrange polynomial interpolating at the $ntot$ points. The weights $d_r(x)$ are defined:

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It is possible to prove that they are unique.

The weights $\omega_r(x)$

Once we have β_r and $d_r(x)$, we define the weights

$$\tilde{\omega}_r(x) = \frac{d_r(x)}{(\epsilon + \beta_r)^2},$$

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The final weights are their normalization:

$$\omega_r(x) = \frac{\tilde{\omega}_r(x)}{\sum_{j=0}^{nlp-1} \tilde{\omega}_r(x)}.$$

Order

The order of WENO-$ntot$,lpo is

$$lpo + 1.$$

Linear Advection

$1D$-linear advection is just

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It is obviously mass-conservative:

$$\int_{\mathbb{R}} f(t,x)dx = \int_{\mathbb{R}} f_0(x-vt)dx = \int_{\mathbb{R}} f_0(x)dx = M.$$

Semi Lagrangian Method

Knowing

$$f(t^n, x_i)$$

the Δt step in time is performed by computing

$$f(t^{n+1}, x_i) = f(t^n, x_i - v\Delta t).$$

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This method is not mass-conservative.

Flux Balance Method

We impose the conservation of the mass this way:

$$\int_{\mathcal{E}_{sx}}^{\mathcal{E}_{dx}} f(t^{n+1},\xi)d\xi = \int_{\mathcal{E}_{sx}}^{\mathcal{E}_{dx}} f(t^n,\xi - v\Delta t)d\xi.$$

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$$\int_{\mathcal{E}_{sx}}^{\mathcal{E}_{dx}} f(t^{n+1}, \xi) d\xi = \int_{\mathcal{E}_{sx}-v\Delta t}^{\mathcal{E}_{dx}-v\Delta t} f(t^n, \xi) d\xi$$

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$$= \int_{\mathcal{E}_{sx}-v\Delta t}^{\mathcal{E}_{sx}} f(t^n,\xi)d\xi + \int_{\mathcal{E}_{sx}}^{\mathcal{E}_{dx}} f(t^n,\xi)d\xi - \int_{\mathcal{E}_{dx}-v\Delta t}^{\mathcal{E}_{dx}} f(t^n,\xi)d\xi$$

Use as notation $\Phi(t,x) = \int_{x-v\Delta t}^{x} f(t,\xi)d\xi$.

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$$= \int_{\mathcal{E}_{sx}}^{\mathcal{E}_{dx}} f(t^n, \xi) d\xi + \Phi^n(\mathcal{E}_{sx}) - \Phi^n(\mathcal{E}_{dx})$$

Dividing by $\Delta x = \mathcal{E}_{dx} - \mathcal{E}_{sx}$,

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Use as notation $\Phi(t, x) = \int_{x-v\Delta t}^{x} f(t, \xi) d\xi$. The step in time will be performed this way:

$$f_i^{n+1} = f_i^n + \frac{\Phi^n\left(x_{i-\frac{1}{2}}\right) - \Phi^n\left(x_{i+\frac{1}{2}}\right)}{\Delta x}.$$

Tests

- Total Variation control

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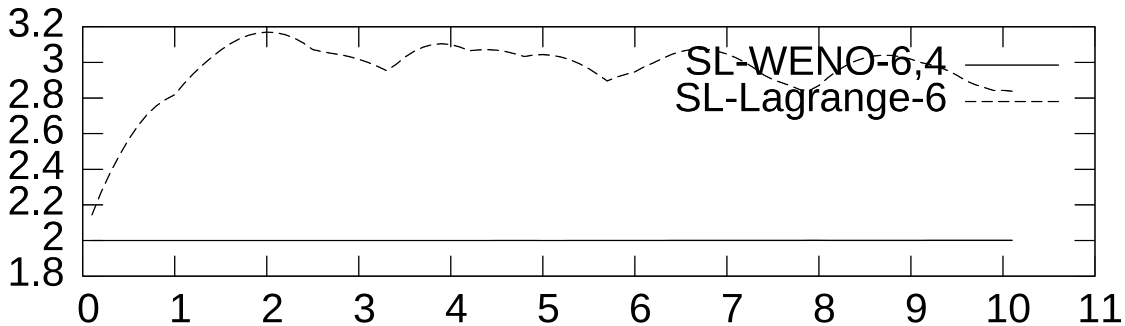
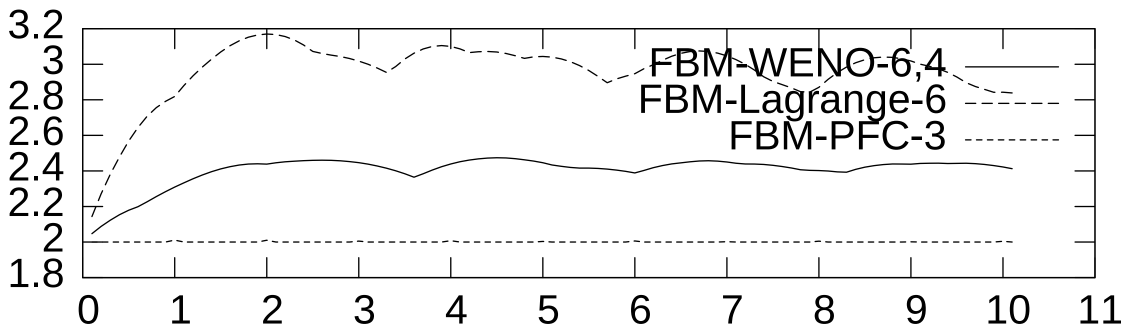
In the exact solution the Total Variation is constant, but in some numerical methods spurious oscillations appear due to high derivatives.

Total Variation Control





Performing PWENO

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- Non-linear $1D$ Landau damping
- Quasi-Relativistic Vlasov-Maxwell

Vlasov-Boltzmann

The model is

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The solution was proven to tend to

$$f_s = M\left(\int_{\mathbb{R}} \exp\left(-\Phi_0(x)\right) dx\right)^{-1} \exp\left(-\Phi_0(x)\right) M_1(v),$$

in L^1-norm, thanks to the bound given by the relative entropies

Vlasov-Boltzmann

$$\begin{cases} H[f;f_s] = \displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f-f_s|^2}{f_s}\,dvdx \\ \tilde{H}[f;\rho M_1] = \displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f-\rho M_1|^2}{f_s}\,dvdx \end{cases}.$$

$$\|f-f_s\|_{L^1}^2 \leq H[f;f_s] \leq C(\epsilon, f_0)t^{-\frac{1}{\epsilon}}.$$

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$$\|f-f_s\|_{L^1}^2 \leq H[f;f_s] \leq C(\epsilon, f_0) t^{-\frac{1}{\epsilon}}.$$

Take as particular case

$$\Phi_0(x) = \frac{x^2}{2}.$$

This produces a rotation of $f_0(x,v)$ and a thermalization of the velocity towards $M_1(v)$.

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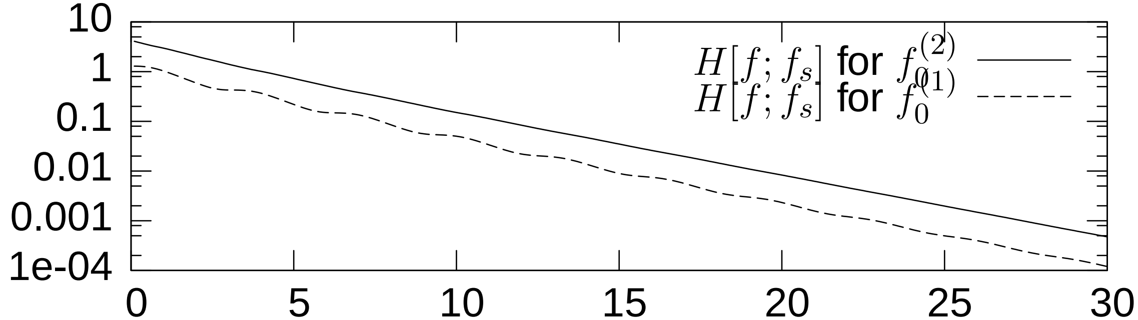

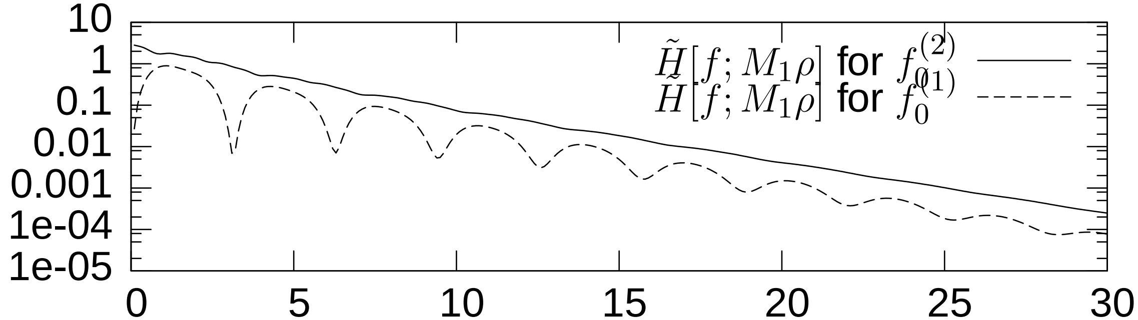

Vlasov-Boltzmann

The decay slope and the oscillation frequency do not depend on the initial datum nor on the x-domain we choose; they are determined by the system itself. A numerical test shows:

$f_0(x)$	L	ω	γ
$f_0^{(1)}(x)$	4π	3.15	-0.298368
$f_0^{(1)}(x)$	6π	3.15	-0.298872
$f_0^{(2)}(x)$	4π	3.125	-0.304400
$f_0^{(2)}(x)$	6π	3.125	-0.304858

Landau damping

Non-collisional Vlasov-Poisson equation with initial datum

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \left[1 + \alpha \cos(kx)\right]$$

produces a transfer of energy from the electric field (potential energy) to the particles (kinetic energy). The decay of the electric energy is oscillating, due to the bounce time.

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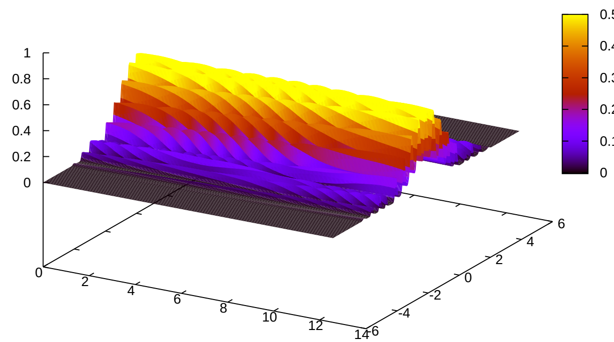
Non-linear Landau damping

Take

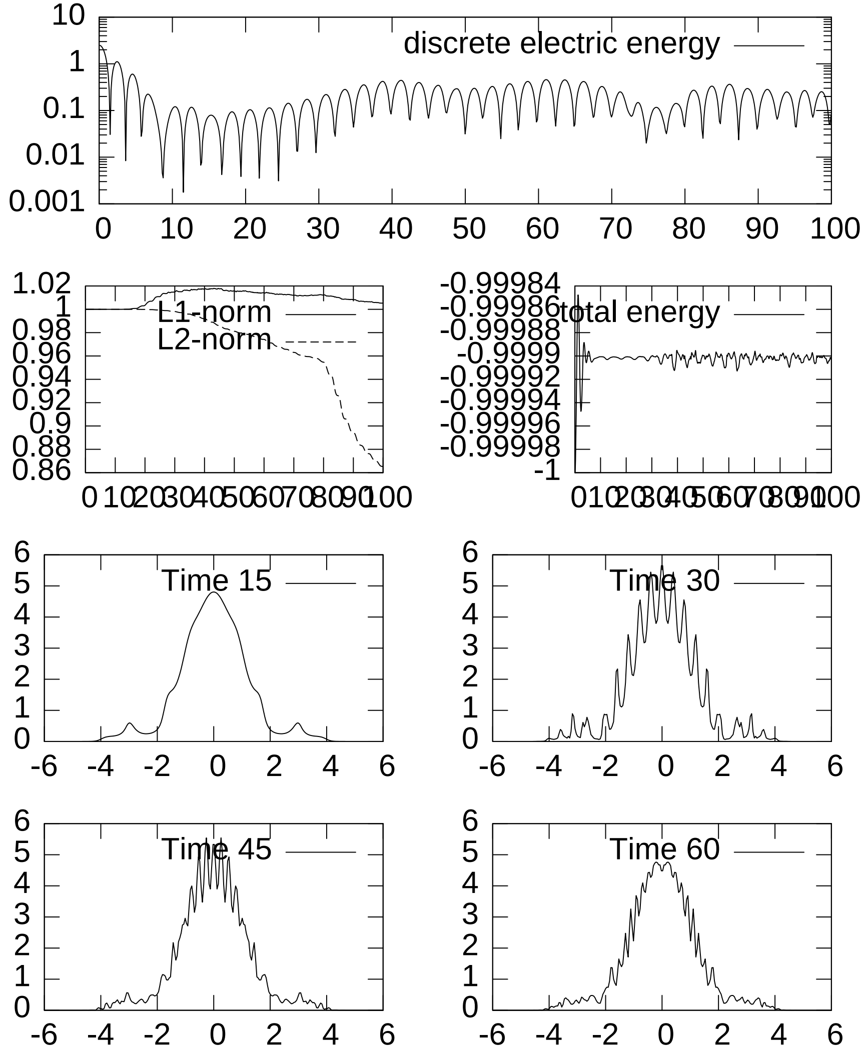
$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \left[1 + 0.5 \cos(0.5x)\right].$$

An interesting phenomenon to observe is the filamentation of the phase space:

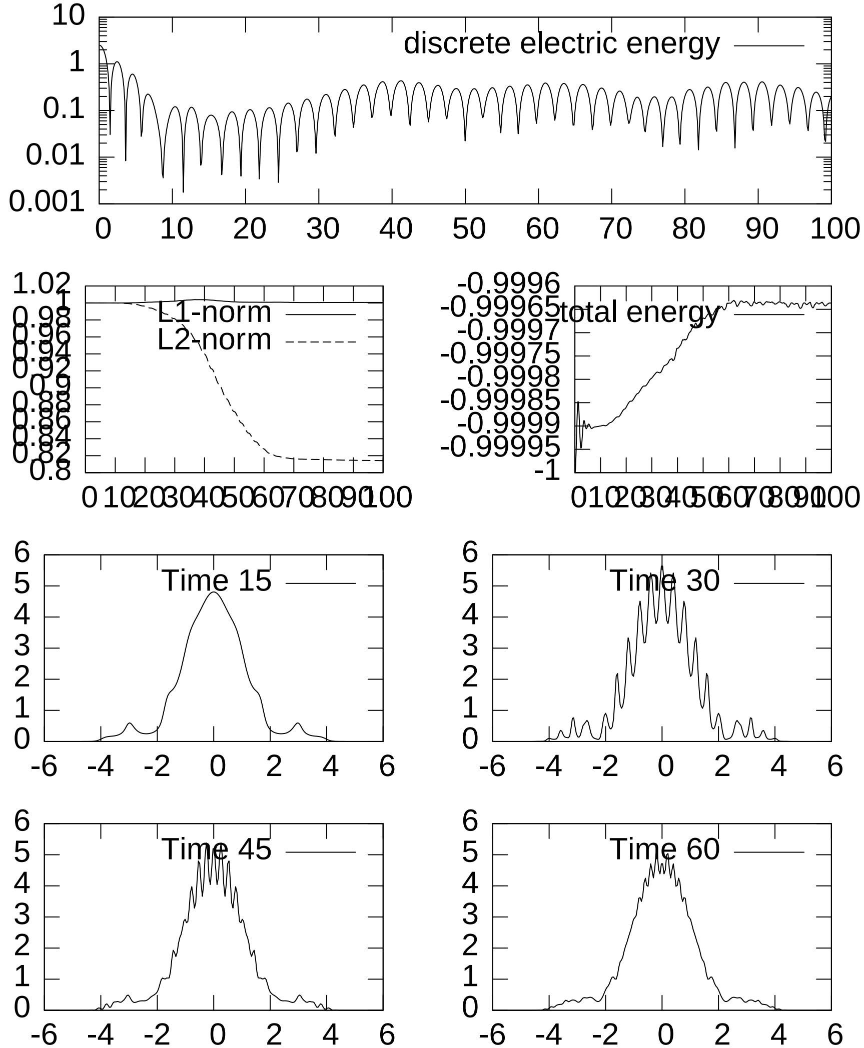


The interpolation method must be good enough to properly stand this phenomenon and not to add spurious oscillations.

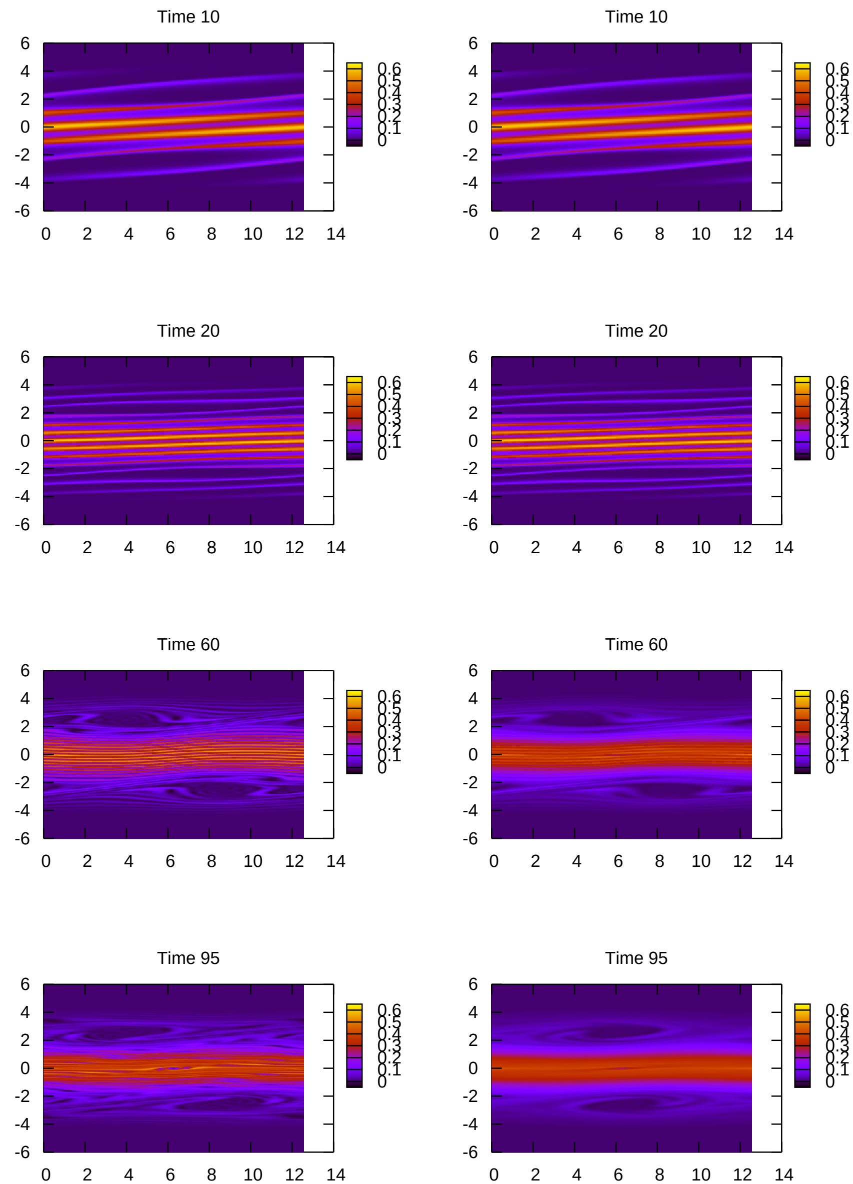
Lagrange reconstruction



PWENO reconstruction



PWENO reconstruction



Vlasov-Maxwell

The model is

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{p}{\sqrt{1+p^2}} \frac{\partial f}{\partial x} - \left(\eta^{-1} E + A \frac{\partial A}{\partial x}\right) \frac{\partial f}{\partial p} = 0 \\ \Phi_{xx} = \eta^{-2} \left[n^{ext} - \rho \right] \\ \frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial x^2} = -\eta^{-2} \rho A \\ \mathcal{E} = -\frac{\partial A}{\partial t} \\ \frac{\partial \mathcal{E}}{\partial x} = -\frac{\partial B}{\partial t} \\ -\frac{\partial B}{\partial x} = -\eta^{-2} A \int_{\mathbb{R}} f \, dp + \frac{\partial \mathcal{E}}{\partial t}. \end{cases}$$

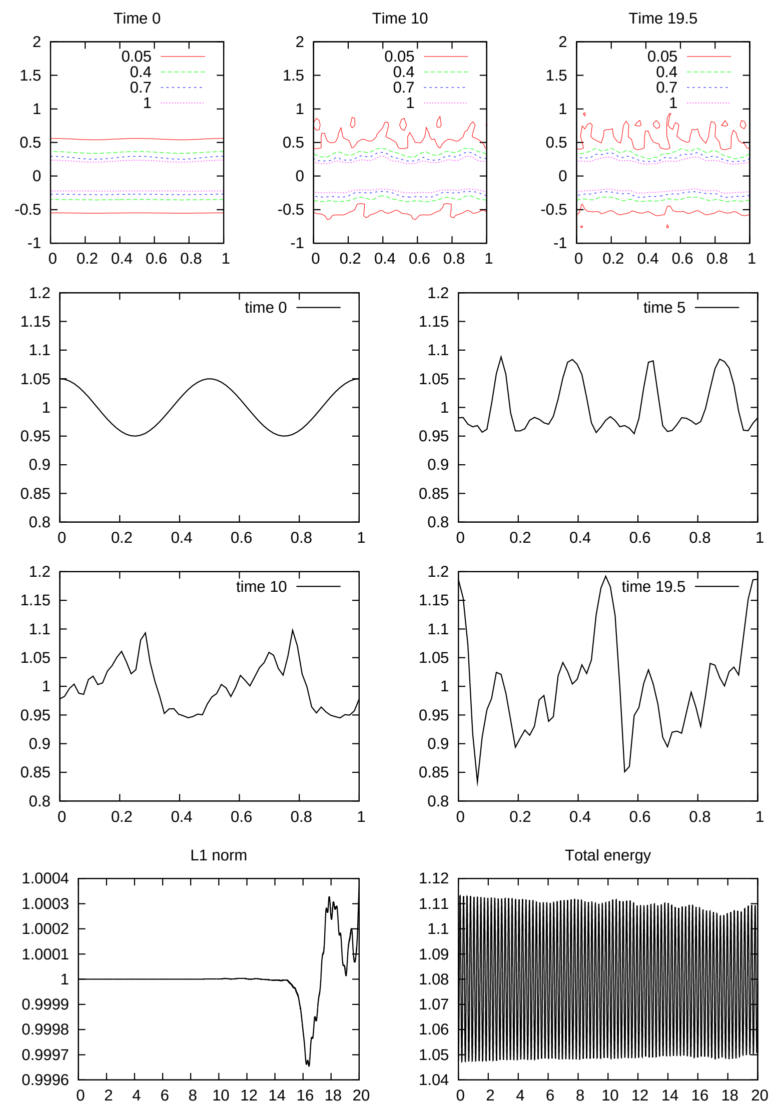
Vlasov-Maxwell

The model is

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{p}{\sqrt{1+p^2}} \frac{\partial f}{\partial x} - \left(\eta^{-1} E + A \frac{\partial A}{\partial x}\right) \frac{\partial f}{\partial p} = 0 \\ \Phi_{xx} = \eta^{-2} \left[n^{ext} - \rho \right] \\ \frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial x^2} = -\eta^{-2} \rho A \\ \mathcal{E} = -\frac{\partial A}{\partial t} \\ \frac{\partial \mathcal{E}}{\partial x} = -\frac{\partial B}{\partial t} \\ -\frac{\partial B}{\partial x} = -\eta^{-2} A \int_{\mathbb{R}} f dp + \frac{\partial \mathcal{E}}{\partial t}. \end{cases}$$

As very violent oscillations are produced, it is important that the interpolation method does not add spurious oscillations.

Lagrange reconstruction



PWENO reconstruction