VLADG

Nicolas Crouseilles, Michel Mehrenberger, Éric Sonnendrücker, Francesco Vecil

Luminy, 19 July - 27 August 2010

Outline

Outline

The guiding-center model

We plan to simulate the guiding-center model through a Discontinuous Galerkin discretization.

The transport equation

$$\frac{\partial f}{\partial t} + \text{div}_{x_1,x_2}(Ef) = 0$$

for $(t, x_1, x_2) \in [0, +\infty[\times[0, 2\pi] \times [0, 2\pi]$, with periodic boundary conditions.

The electric field

The electric field is given by $E = -\nabla_{x_1,x_2}\Phi$, Φ being the potential given by

$$-\Delta_{x_1,x_2}\Phi = f,$$

with periodic boundary conditions.

Outline

Linear advection

The first steps towards the goal are the solution and testing of the linear advection problems:

The 1D linear advection

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} = 0,$$

with periodic boundary conditions.

The 2D linear advection

$$\frac{\partial f}{\partial t} + a_1\frac{\partial f}{\partial x_1} + a_2\frac{\partial f}{\partial x_2} = 0,$$

with periodic boundary conditions.

Landau damping

A good benchmark for the 2D linear advection is the simulation of the Landau damping, which requires the coupling of the transport equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial v} = 0,$$

to the Poisson equation for the computation of the force field

$$-\frac{d^2\Phi}{dx^2} = 1 - \int f dv,$$

where the border conditions are taken periodic for both the transport equation and the Poisson equation.

Nonlinear advection

The next steps are the solution and testing of the nonlinear advection problems:

The 1D nonlinear advection

$$\frac{\partial f}{\partial t} + \frac{\partial (af)}{\partial x} = 0,$$

with periodic boundary conditions.

The 2D nonlinear advection

$$\frac{\partial f}{\partial t} + \frac{\partial (a_1 f)}{\partial x_1} + \frac{\partial (a_2 f)}{\partial x_2} = 0.$$

with periodic boundary conditions.

Outline

Discretization

Partition of the computational domain

The computational domain $\Omega = [0, 1]$ is partitioned into N cells of size Δx:

$$\Omega = \bigcup_{i=0}^{N-1} I_i, \qquad I_i = [x_{i-1/2}, x_{i+1/2}].$$

Discontinuous Galerkin space

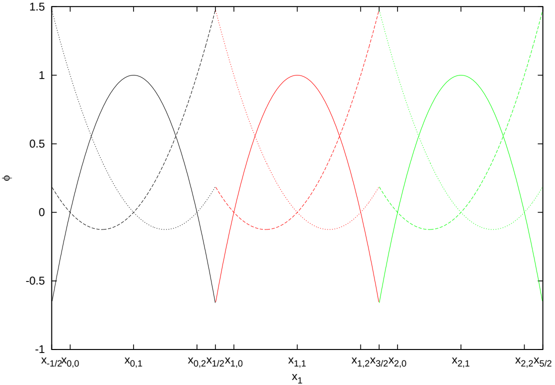
Let V^d the discontinuous finite elements space:

$$V^d = \left\{ \psi \in L^2(\Omega) : \psi \in \mathbb{R}_d[X](I_i), \qquad i = 0, \ldots, N-1 \right\}.$$

Choice of the basis

Lagrange polynomials

We choose to use the Lagrange polynomials at the Gauß points as basis.



Choice of the basis

The Gauß points on the interval $[-1, 1]$

The Gauß points $\{\alpha_r\}_{r=0}^{d}$ and the Gauß weights $\{\omega_r\}_{r=0}^{d}$ are quadrature points determined by imposing

$$\int_{-1}^{1} f(x)dx = \sum_{r=0}^{d} \omega_r f(\alpha_r)$$

for all polynomials $f \in \mathbb{R}_{2d+1}[X]$.

Distributing the Gauß points

We can now introduce the notation $x_{i,j}$ for the j-th Gauß point inside the interval I_i; more precisely

$$x_{i,j} = x_{i-1/2} + \frac{\Delta x}{2}\alpha_j.$$

Choice of the basis

Orthogonality of the basis

As the Lagrange polynomials at the Gauß points are defined by

$$\varphi_{i,j} = \prod_{l=0,l\neq j}^{d} \frac{x - x_{i,l}}{x_{i,j} - x_{i,l}},$$

it is easy to check that

$$\int_{I_i} \varphi_{i,j_1}(x)\varphi_{i,j_2}(x) = \frac{\Delta x}{2} \sum_{r=0}^{d} \omega_r \varphi_{i,j_1}(\alpha_r)\varphi_{i,j_2}(\alpha_r) = \frac{\Delta x}{2}\omega_{j_1}\delta_{j_1,j_2}.$$

Notation for the future

We shall denote by $\{\tilde{\varphi}^j\}_{j=0}^d$ and $\{\tilde{\alpha}_j\}_{j=0}^d$ the Lagrange polynomials and the Gauß points on the interval $[0, 1]$.

Characteristics-based method

Starting point

Test f^{n+1} over the interval I_i:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f^{n+1}(x)\varphi(x)dx$$

use the solution given by the characteristics

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f^{n+1}(x)\varphi(x)dx = \int_{x_{i-1/2}}^{x_{i+1/2}} f^{n}(x-a\Delta t)\varphi(x)dx$$

change variables $x \to x - a\Delta t$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f^{n+1}(x)\varphi(x)dx = \int_{x_{i-1/2}-a\Delta t}^{x_{i+1/2}-a\Delta t} f^{n}(x)\varphi(x+a\Delta t)dx.$$

Characteristics-based method

Representation of $f(x)$

The representation of f in the DG basis is $f(x) \approx \sum_{i'=0}^{N-1} \sum_{j'=0}^{d} f_{i',j'} \varphi_{i',j'}(x)$.

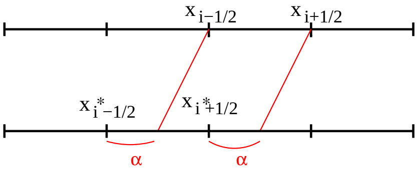
Developing the scheme

Inject the representation of $f(x)$ into the scheme $\int_{x_{i-1/2}}^{x_{i+1/2}} f^{n+1}(x)\varphi(x)dx = \int_{x_{i-1/2}-a\Delta t}^{x_{i+1/2}-a\Delta t} f^n(x)\varphi(x+a\Delta t)dx$ and test on $\varphi_{i,j}(x)$:

$$f_{i,j}^{n+1} \frac{\Delta x}{2} \omega_j = \sum_{i',j'} f_{i',j'}^n \int_{x_{i-1/2}-a\Delta t}^{x_{i+1/2}-a\Delta t} \varphi_{i',j'}(x)\varphi_{i,j}(x+a\Delta t).$$

Characteristics-based method

Treating the right hand side



$$
\begin{aligned}
& \int_{x_{i-1/2}-a\Delta t}^{x_{i+1/2}-a\Delta t} \varphi_{i',j'}(x)\varphi_{i,j}(x+a\Delta t) \\
= & \int_{x_{i^*-1/2}+\alpha\Delta x}^{x_{i^*+1/2}+\alpha\Delta x} \varphi_{i',j'}(x)\varphi_{i,j}(x+a\Delta t) \\
= & \int_{x_{i^*-1/2}+\alpha\Delta x}^{x_{i^*+1/2}} \varphi_{i',j'}(x)\varphi_{i,j}(x+a\Delta t) + \int_{x_{i^*+1/2}}^{x_{i^*+1/2}+\alpha\Delta x} \varphi_{i',j'}(x)\varphi_{i,j}(x+a\Delta t)
\end{aligned}
$$

Characteristics-based method

Changing variables

We change variables (and divide by $\frac{\Delta x}{2}$) in order to reduce to integrating on $[0,1]$:

$$\begin{aligned} f_{i,j}^{n+1} &= \frac{1}{\omega_j}\sum_{j'} f^n_{i^*,j'}(1-\alpha)\int_{u=0}^{1}\tilde{\varphi}^{j'}(\alpha+u(1-\alpha))\tilde{\varphi}^{j}(u(1-\alpha))du \\ &+ \frac{1}{\omega_j}\sum_{j'} f^n_{i^*+1,j'}\alpha\int_{u=0}^{1}\tilde{\varphi}^{j'}(\alpha u)\tilde{\varphi}^{j}(\alpha(u-1)+1)du \end{aligned}$$

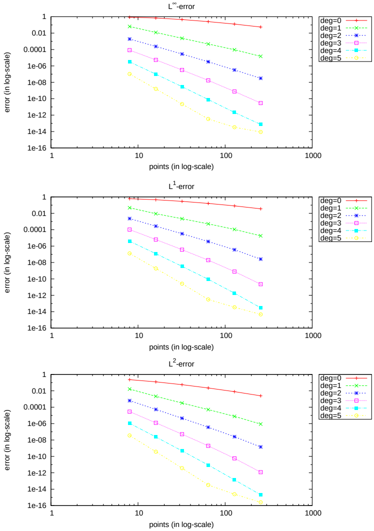
Gauß quadrature

Finally we integrate using the Gauß quadrature:

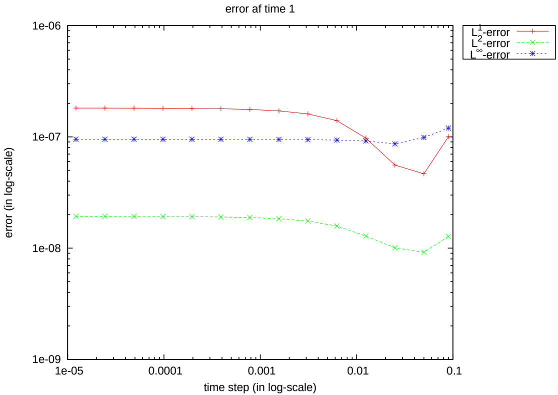
$$\begin{aligned} f_{i,j}^{n+1} &= \frac{1}{\omega_j}\sum_{j'} f^n_{i^*,j'}(1-\alpha)\sum_{r=0}^{d}\omega_r\tilde{\varphi}^{j'}(\alpha+\tilde{\alpha}_r(1-\alpha))\tilde{\varphi}^{j}(\tilde{\alpha}_r(1-\alpha)) \\ &+ \frac{1}{\omega_j}\sum_{j'} f^n_{i^*+1,j'}\alpha\sum_{r=0}^{d}\omega_r\tilde{\varphi}^{j'}(\alpha\tilde{\alpha}_r)\tilde{\varphi}^{j}(\alpha(\tilde{\alpha}_r-1)+1). \end{aligned}$$

Outline

Order in space



Order in time



Outline

Discretization

Partition of the computational domain

The computational domain $\Omega = [0,1] \times [0,1]$ is partitioned into $N_{x_1} \times N_{x_2}$ cells of size $\Delta x_1 \times \Delta x_2$:

$$\Omega = \bigcup_{i,k} I_i \times J_k, \qquad I_i = [(x_1)_{i-1/2}, (x_1)_{i+1/2}], \qquad J_k = [(x_2)_{k-1/2}, (x_2)_{k+1/2}].$$

Discontinuous Galerkin space

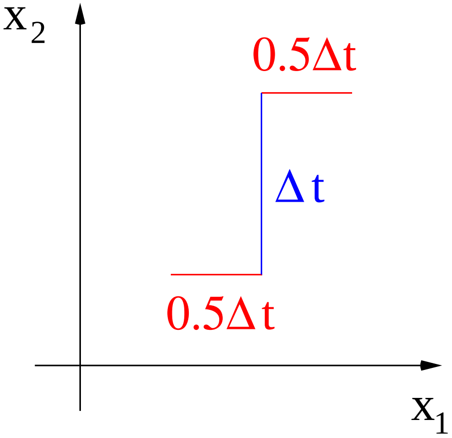
Let V^d be the discontinuous finite elements space as tensor product of the spaces for each variable:

$$V^d = \left\{ v \in L^2(\Omega) : v(x_1, x_2) = \varphi(x_1)\psi(x_2), \varphi \in \mathbb{R}_d[X](I_i), \psi \in \mathbb{R}_d[X](J_k) \right\}.$$

Discretization

Time discretization

Instead of solving the true 2D problem, we split the (x_1, x_2)-domain into advection along x_1 and advection along x_2 using Strang splitting:



Outline

Landau damping

A 2D linear advection

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial v} = 0$$

is coupled to a Poisson equation for the computation of the electric field:

$$\frac{\partial E}{\partial x} = \rho - 1.$$

Landau damping

Integration of the Poisson equation

Use Green kernel for the Poisson equation to obtain

$$E(x) = \frac{1}{L}\int_0^L y\rho(y)dy - \int_x^L \rho(y)dy + \frac{L}{2} - x.$$

Inject the representations of $E(x) \approx \sum_{i,j} E_{i,j}\varphi_{i,j}(x)$ and $\rho(x) \approx \sum_{i,j} \rho_{i,j}\varphi_{i,j}(x)$, use orthogonality of the basis, the same changes of variables as before and quadrature formulae to get

$$\begin{aligned} E_{i,j} \quad &\approx \quad \frac{\Delta x_1}{2L}\sum_{i',j'} x_{i',j'}\omega_{j'}\rho_{i',j'} - \frac{\Delta x_1}{2}(1-\tilde{\alpha}_j)\sum_{j'}\rho_{i,j'}\sum_{r=0}^{d}\omega_r\tilde{\varphi}^{j'}(\tilde{\alpha}_j + \tilde{\alpha}_r(1-\tilde{\alpha}_j)) \\ &\quad -\frac{\Delta x_1}{2}\sum_{i'=i+1}^{N-1}\sum_{j'=0}^{d}\omega_{j'}\rho_{i',j'} + \frac{L}{2} - x_{i,j}. \end{aligned}$$

Landau damping

Period and decay

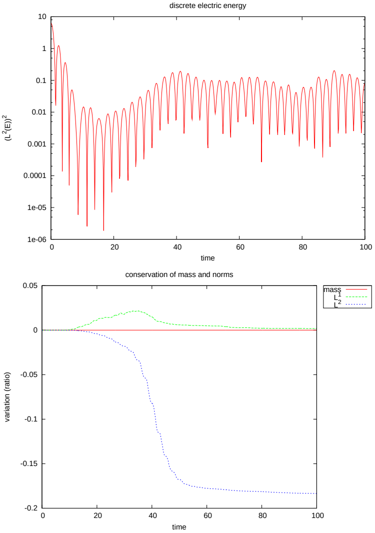
Set the problem $f_0(x,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} (1 + \alpha \cos(kx))$.

k	$\alpha = 0.001$ (linear)	$\alpha = 0.5$ (nonlinear)
0.2	$\pm 1.07154 + 6.81267 \times 10^{-5} i$ $(\pm 1.0640 - 5.51 \times 10^{-5} i)$	$\pm 1.09402 - 0.00107607i$
0.3	$\pm 1.16209 - 0.0124224i$ $(\pm 1.1598 - 0.0126i)$	$\pm 1.30507 - 0.128511i$
0.4	$\pm 1.28645 - 0.0659432i$ $(\pm 1.2850 - 0.0661i)$	$\pm 1.3581 - 0.205133i$
0.5	$\pm 1.41696 - 0.152849i$ $(\pm 1.4156 - 0.1533i)$	$\pm 1.47343 - 0.279512i$

Table: **1D Landau damping.** The decay rate and period of the oscillations of the electric field in the Landau damping problem. Here, $d = 4$, $N_x \times N_v = 30 \times 30$.

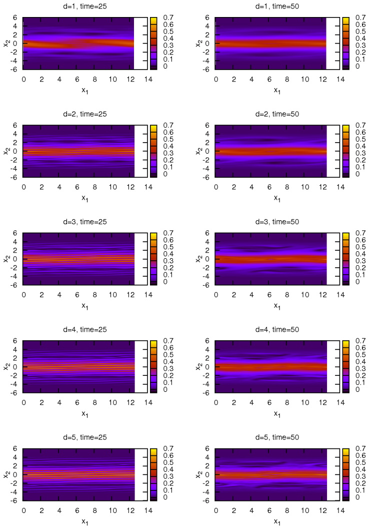
Landau damping

Nonlinear Landau damping



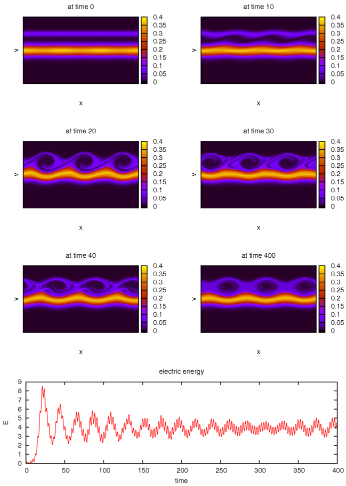
Landau damping

Filamentation of the phase-space



Bump-On-Tail

By using $f_0(x,v) = \frac{9}{10\sqrt{2\pi}} e^{-\frac{v^2}{2}} + \frac{2}{10\sqrt{2\pi}} e^{-2|v-4.5|^2}(1 + 0.03\cos(0.3x))$ as initial condition, we expect to observe some vortices.



Outline

Characteristics-based method

The strategy follows that of the 1D linear advection.

Starting point

Test f^{n+1} over the interval I_i:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f^{n+1}(x)\varphi(x)dx$$

use the solution given by the characteristics

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f^{n+1}(x)\varphi(x)dx = \int_{x_{i-1/2}}^{x_{i+1/2}} f^{n}(\mathcal{X}(t^n;t^{n+1},x))J(t^n;t^{n+1},x)\varphi(x)dx$$

change variables $x \to \mathcal{X}(t^n;t^{n+1},x)$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f(t^{n+1},x)\varphi(x)dx = \int_{\mathcal{X}(t^n;t^{n+1},x_{i-1/2})}^{\mathcal{X}(t^n;t^{n+1},x_{i+1/2})} f(t^n,x)\varphi(\mathcal{X}(t^{n+1};t^n,x))dx.$$

Characteristics-based method

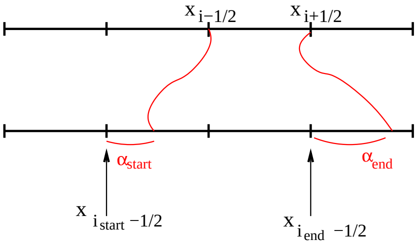
Developing the scheme

Inject the representation of $f(x)$ into the scheme and test on $\varphi_{i,j}(x)$:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f(t^{n+1},x)\varphi_{i,j}(x)dx = \sum_{i',j'} f^n_{i',j'} \int_{\mathcal{X}(t^n;t^{n+1},x_{i-1/2})}^{\mathcal{X}(t^n;t^{n+1},x_{i+1/2})} \varphi_{i',j'}(x)\varphi_{i,j}(\mathcal{X}(t^{n+1};t^n,x))dx.$$

Some notations

Let $i_{start} = i_{start}(i)$, $\alpha_{start} = \alpha_{start}(i) \in [0,1]$ and $i_{start} = i_{end}(i)$, $\alpha_{end} = \alpha_{end}(i) \in [0,1]$ such that



Characteristics-based method

Treating the right hand side

The integral is decomposed into three pieces:

$$\begin{aligned} f_{i,j}^{n+1}\omega_j\frac{\Delta x}{2} &= \sum_{i',j'} f_{i',j'}^{n}\int_{x_{i_{start}-1/2}+\alpha_{start}\Delta x}^{x_{i_{start}+1/2}} \varphi_{i',j'}(x)\varphi_{i,j}(\mathcal{X}(t^{n+1};t^n,x))dx \\ &+ \sum_{i',j'} f_{i',j'}^{n}\sum_{i''=i_{start}+1}^{i_{end}-1}\int_{x_{i''-1/2}}^{x_{i''+1/2}} \varphi_{i',j'}(x)\varphi_{i,j}(\mathcal{X}(t^{n+1};t^n,x))dx \\ &+ \sum_{i',j'} f_{i',j'}^{n}\int_{x_{i_{end}-1/2}}^{x_{i_{end}-1/2}+\alpha_{end}\Delta x} \varphi_{i',j'}(x)\varphi_{i,j}(\mathcal{X}(t^{n+1};t^n,x))dx. \end{aligned}$$

Characteristics-based method

The scheme

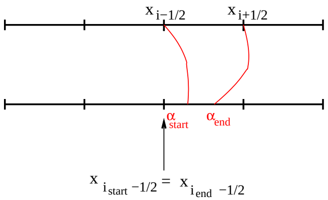
By changing variables to reduce to integrating on $[0,1]$ and using Gauß quadrature (the same strategy as before) we are led to

$$
\begin{aligned}
f_{i,j}^{n+1} \quad = \quad & \frac{1}{\omega_j} \sum_{j'=0}^{d} f_{i_{start},j'}^{n} (1-\alpha_{start}) \sum_{r=0}^{d} \omega_r \tilde{\varphi}^{j'} (\alpha_{start} + \tilde{\alpha}_r (1-\alpha_{start})) \\
& \times \varphi_{i,j}(\mathcal{X}(t^{n+1}; t^n, x_{i_{start}-1/2} + (\alpha_{start} + \tilde{\alpha}_r(1-\alpha_{start}))\Delta x)) \\
+ \quad & \frac{1}{\omega_j} \sum_{i''=i_{start}+1}^{i_{end}-1} \sum_{j'=0}^{d} f_{i'',j'}^{n} \sum_{r=0}^{d} \omega_r \tilde{\varphi}^{j'}(\tilde{\alpha}_r) \varphi_{i,j}(\mathcal{X}(t^{n+1}; t^n, x_{i''-1/2} + \tilde{\alpha}_r \Delta x)) \\
+ \quad & \frac{1}{\omega_j} \sum_{j'=0}^{d} f_{i_{end},j'}^{n} \alpha_{end} \sum_{r=0}^{d} \omega_r \tilde{\varphi}^{j'}(\alpha_{end}\tilde{\alpha}_r) \varphi_{i,j}(\mathcal{X}(t^{n+1}; t^n, x_{i_{end}-1/2} + \alpha_{end}\tilde{\alpha}_r \Delta x)).
\end{aligned}
$$

Characteristics-based method

Case of compression

In case a compression should happen



then the formula reduces to just one integral:

$$f_{i,j}^{n+1} = \frac{1}{\omega_j}\sum_{j'=0}^{d} f_{i_{start},j'}^{n}(\alpha_{end}-\alpha_{start})\sum_{r=0}^{d}\omega_r\varphi^{j'}((\alpha_{end}-\alpha_{start})\tilde{\alpha}_r+\alpha_{start})$$
$$\times\varphi_{i,j}(\mathcal{X}(t^{n+1};t^n,x_{i_{start}-1/2}+\Delta x((\alpha_{end}-\alpha_{start})\tilde{\alpha}_r+\alpha_{start}))).$$

Characteristics-based method

Solving the characteristics

In order to write the scheme, we still need to solve the characteristics, both forward and backward. In order to do this, we shall use an explicit formula if it is available, otherwise Runge-Kutta methods of order 1 to 4.

Outline

Benchmark test

We take as test case

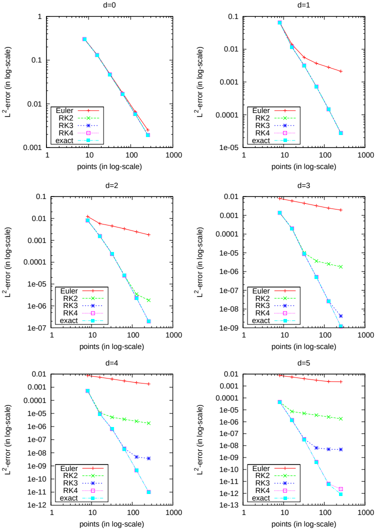
$$\frac{\partial f}{\partial t} + \frac{\partial (sin(x) f)}{\partial x} = 0,$$

which has explicit characteristics and solution:

$$\begin{aligned} \mathcal{X}(s;t,x) &= 2\arctan\left(\tan\left(\frac{x}{2}\right)e^{s-t}\right) + 2\pi\left\lfloor\frac{x+\pi}{2\pi}\right\rfloor \\ f(t,x) &= \frac{1}{1+\left(\tan\left(\frac{x}{2}\right)e^{-t}\right)^2}\frac{1}{cos^2\left(\frac{x}{2}\right)}e^{-t}. \end{aligned}$$

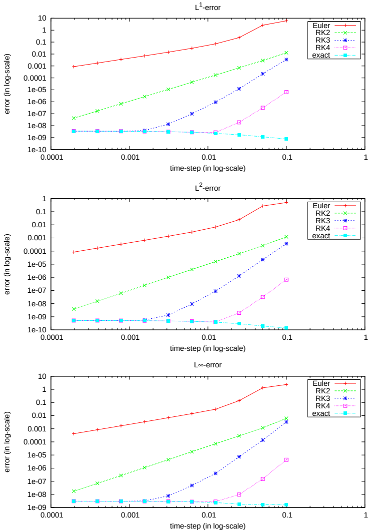
Benchmark test

Order in space



Benchmark test

Order in time



Outline

Splitting strategy

As well as the 1D linear advection solver could be exploited to solve 2D linear advection through Strang splitting, the 2D nonlinear advection can be solved by splitting the (x_1, x_2)-space.

The main problem which is still work in progress is the solution of the potential for the guiding-center problem, namely solving

$$-\Delta_{x_1,x_2}\Phi = f.$$

The strategy which we are following consists in switching to the Fourier-space, solving the electric field there and anti-transform to obtain it in the (x_1, x_2)-space.