AP schemes for intermediate models between a kinetic equation and its diffusive limit

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Outline

1. Introduction
   - Motivation
   - Setting
2. Approximations
   - Heat equation and $\mathbb{P}^1$-approximation
   - Intermediate models
3. Asymptotic-preserving schemes
   - Kinetic equation
   - First-order closure
   - Zeroth-order closure
4. Experiments
   - AP properties of the schemes
   - Comparisons
   - Su-Olson tests

Outline

Kinetic equations

Origin.

Kinetic equations arise in physics and engineering when a huge amount of particles is described statistically by a distribution function $f(t,x,v)$. Some examples:

- semiconductor physics;
- gas dynamics;
- plasma physics;
- collective behaviour models.

Diffusive scaling.

The diffusive scaling is meant to represent a regime in which the mean free path travelled by the particles goes to zero.

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Setting the problem

Kinetic equation.

Take the 1D **transport equation**

$$\varepsilon\partial_t f_\varepsilon + v\partial_x f_\varepsilon = \frac{1}{\varepsilon}\mathcal{Q}[f_\varepsilon], \qquad \mathcal{Q}[f_\varepsilon] = \langle f_\varepsilon\rangle - f_\varepsilon$$

with $(t, x, v) \in [0, T] \times \mathbb{R} \times V$, completed by initial and boundary conditions. Particles are not driven by any force field and interact through a relaxation-type collision operator.

Setting the problem

The velocity space (V, μ)

V is a space endowed with a measure μ such that it satisfies:

(i) $\langle 1 \rangle = 1$;

(ii) $\langle h(v) \rangle = 0$ for any odd function h;

(iii) $\langle v^2 \rangle = d \in \mathbb{R}_{>0}$.

In our notations $\langle f \rangle = \int_V f(v) d\mu(v)$.

Examples of (V, μ)

- $V = (-1, 1), \qquad d\mu(v) = \frac{1}{2} d\lambda(v)$;
- $V = (-1, 1), \qquad \{v_i\}_{i=1}^N \subseteq (-1, 1)^N, \qquad d\mu(v) = \frac{1}{N} \sum_{i=1}^N \delta(v = v_i)$: the points $\{v_i\}_{i=1}^N$ have to be well chosen, otherwise properties **(ii)** and **(iii)** do not hold;
- $V = \mathbb{R}, \qquad d\mu(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} d\lambda(v)$.

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Heat equation

Diffusive limit.

As $\varepsilon \to 0$, f_ε relaxes to F_0 solution to the **heat equation**:

$$\varepsilon\partial_t f_\varepsilon + v\partial_x f_\varepsilon = \frac{1}{\varepsilon}\mathcal{Q}[f_\varepsilon] \quad \xrightarrow[\varepsilon\to 0]{} \quad \partial_t F_0 - \left\langle v^2 \right\rangle \partial_{xx}^2 F_0 = 0.$$

Proof.

Formally take the Hilbert expansion in ε

$$f_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \dots$$

inject into the kinetic equation and extract the F_i.

Drawbacks.

- The heat equation is not v-dependent: no microscopic feature.
- The heat equation transports information at infinite velocity, the transport equation at $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ velocity.

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Approximations

The $\mathbb{P}1$-approximation

By truncating the Hilbert expansion at first order

$$f_\varepsilon \approx F_0 + \varepsilon F_1$$

we obtain the $\mathbb{P}1$-approximation

$$f_\varepsilon(t,x,v) \approx F_0(t,x) - \varepsilon v \partial_x F_0(t,x)$$

which is v-dependent, so that it somehow restores some microscopic features.

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- The $\mathbb{P}1$-approximation might be negative.
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Moments

Define the zeroth, first and second order moments by

$$\begin{pmatrix} \rho \\ J \\ \mathbb{P} \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ v/\varepsilon \\ v^2 \end{pmatrix} f_\varepsilon \right\rangle .$$

Moment equations

Integrating the kinetic equation, we obtain the moment equations

$$\begin{aligned} \partial_t \rho + \partial_x J &= 0 \\ \varepsilon^2 \partial_t J + \partial_x \mathbb{P} &= -J, \end{aligned}$$

which need some **closure strategy**, the k^{th}-moment equation being dependent on the $(k+1)^{th}$-moment.

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Zeroth-order closure

Two closures are proposed, one at zeroth order and one at first order.

Zeroth order closure

By truncating the modified Hilbert expansion $f_\varepsilon = e^{a_0+\varepsilon a_1+\varepsilon^2 a_2+\cdots}$ at first order

$$f_\varepsilon \approx \exp\left(a_0 + \varepsilon a_1\right)$$

and injecting the approximation thus obtained

$$f_\varepsilon(t,x,v) \approx \frac{\rho(t,x)}{Z(t,x)} e^{-\varepsilon v \frac{\partial_x \rho}{\rho}(t,x)}$$

into the zeroth moment equation, we obtain the following system:

$$\partial_t \rho - \partial_x \left[\frac{\rho}{\varepsilon} \mathbb{G}\left(\varepsilon \frac{\partial_x \rho}{\rho}\right)\right] = 0.$$

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Some notations

We have introduced:

- $Z(t,x)$ is a normalizing factor such that $\langle f_\varepsilon \rangle = \rho(t,x)$;
- $\mathbb{F}(x) = \langle e^{xv} \rangle$;
- $\mathbb{G}(x) = \frac{\mathbb{F}'}{\mathbb{F}}(x)$.

Examples

If $V = (-1,1)$ and $d\mu = \frac{1}{2} d\lambda$ (normalized Lebesgue measure), then

$$\mathbb{G}(x) = \coth(x) - \frac{1}{x}.$$

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First-order closure

Entropy Minimization Principle

The first-order closure comes from the following Entropy Minimization Principle:

$$f_\varepsilon = \operatorname{argmin}\left\{\langle f_\varepsilon \log(f_\varepsilon)\rangle\right\}$$

under the constraints

$$\left\langle \begin{pmatrix} 1 \\ v/\varepsilon \end{pmatrix} f_\varepsilon \right\rangle = \begin{pmatrix} \rho \\ J \end{pmatrix}.$$

The closed system

We can thus express the second moment as

$$\mathbb{P} = \rho\psi\left(\frac{\varepsilon J}{\rho}\right), \qquad \psi(x) = \frac{\mathbb{F}''}{\mathbb{F}}\left(\mathbb{G}^{(-1)}(x)\right),$$

so that the first-order closure reads

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First-order closure

Reconstruction

The microscopic approximation is reconstructed by

$$\tilde{f}_\varepsilon(t,x,v) = \rho(t,x)\frac{\exp\left[v\mathbb{G}^{(-1)}\left(\frac{\varepsilon J}{\rho(t,x)}\right)\right]}{\mathbb{F}\circ\mathbb{G}^{(-1)}\left(\frac{\varepsilon J}{\rho(t,x)}\right)}.$$

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Example

In case $V = (-1,1)$ and $d\mu = \frac{1}{2}d\lambda$ (normalized Lebesgue measure), we have

$$\mathbb{F}(x) = \frac{\sinh(x)}{x}, \qquad \mathbb{G}(x) = \coth(x) - \frac{1}{x}.$$

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Kinetic equation

We propose a splitting scheme for solving the kinetic equation

$$\varepsilon \partial_t f_\varepsilon + v \partial_x f_\varepsilon = \frac{1}{\varepsilon} \left(\langle f_\varepsilon \rangle - f_\varepsilon \right)$$

without need of mesh-resolving parameter ε as it tends to zero.

Decomposition

Split f_ε into its mean value plus fluctuations:

$$f_\varepsilon = \rho_\varepsilon + \varepsilon g_\varepsilon = \langle f_\varepsilon \rangle + \varepsilon g_\varepsilon.$$

Boundedness of the fluctuations

We have from the boundendess in the $L^p(V, \mu)$-spaces of the collision operator

$$\begin{aligned} \|g_\varepsilon\|^2_{L^2_{t,x,v}} &= \int_{\mathbb{R}_{\geq 0}} \int_{\mathbb{R}} \int_V |f_\varepsilon - \rho_\varepsilon|^2 \, dt \, dx \, dv \\ &= \int_{\mathbb{R}_{\geq 0}} \int_{\mathbb{R}} \int_V |\mathcal{Q}[f_\varepsilon]|^2 \, dt \, dx \, dv \\ &\leq C \int_{\mathbb{R}_{\geq 0}} \int_{\mathbb{R}} \int_V |f_\varepsilon|^2 dt \, dx \, dv. \end{aligned}$$

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Kinetic equation

Reformulation

Inject the decomposition $f_\varepsilon = \rho_\varepsilon + \varepsilon g_\varepsilon$ into the kinetic equation to obtain

$$\partial_t f_\varepsilon + \frac{v}{\varepsilon}\partial_x \rho_\varepsilon + v\partial_x g_\varepsilon = \frac{1}{\varepsilon^2}\left(\rho_\varepsilon - f_\varepsilon\right).$$

First-order splitting strategy

(i) solve for a Δt-step the equation $\partial_t f_\varepsilon = \frac{1}{\varepsilon^2}\left(\rho_\varepsilon - f_\varepsilon\right) - \frac{v}{\varepsilon}\partial_x \rho_\varepsilon$;

(ii) solve for a Δt-step the equation $\partial_t f_\varepsilon + v\partial_x g_\varepsilon = 0$.

Evolution of the fluctuations

To complete the scheme, we still need to write the evolution equation for the fluctuations g_ε:

(i) $\langle v \rangle = 0 \Longrightarrow \partial_t \rho_\varepsilon = 0 \qquad \Longrightarrow \qquad \partial_t g_\varepsilon = -\frac{1}{\varepsilon^2} g_\varepsilon - \frac{v}{\varepsilon^2}\partial_x \rho_\varepsilon$;

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Kinetic equation

Résumé

If we take into account only the leading contribution in ε,

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Time discretization of the scheme

Dropping ε-subscript for convenience,

Step (i)a relax f: $f^{n+1/2} = e^{-\frac{\Delta t}{\varepsilon^2}} f^n + \left(1 - e^{-\frac{\Delta t}{\varepsilon^2}}\right)\rho^n;$

Step (i)b relax g: $g^{n+1/2} = e^{-\frac{\Delta t}{\varepsilon^2}} g^n - \left(1 - e^{-\frac{\Delta t}{\varepsilon^2}}\right) v\partial_x \rho^n;$

Step (i)c $\rho^{n+1/2} = \rho^n;$

Step (ii)a convect f: $f^{n+1} = f^{n+1/2} - \Delta t \cdot v\partial_x g^{n+1/2};$

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Kinetic equation

Résumé

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Kinetic equation

AP property

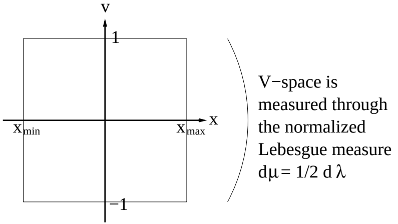
The relaxed scheme $\varepsilon \to 0$ reads:

- $\partial_t f - v^2 \partial^2_{xx} \rho = 0$, which implies that $\partial_t \rho - \langle v^2 \rangle \partial^2_{xx} \rho = 0$. This is the heat equation with the proper constant;
- $g^{n+1/2} = -v \partial_x \rho^{n+1/2}$, which is coherent with the Hilbert expansion.

The scheme, therefore, relaxes to a solver to the proper heat equation.

The case of normalized Lebesgue measure

Let us use in the following the Lebesgue setting:



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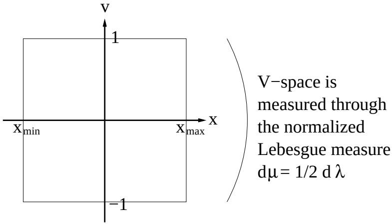
The scheme, therefore, relaxes to a solver to the proper heat equation.

The case of normalized Lebesgue measure

Let us use in the following the Lebesgue setting:



Kinetic equation

Time-space discretized scheme

Step (i)a $f_{i,j}^{n+1/2} = e^{-\frac{\Delta t}{\varepsilon^2}} f_{i,j}^n + \left(1 - e^{-\frac{\Delta t}{\varepsilon^2}}\right) \rho_i^n;$

Step (i)b $g_{i,j}^{n+1/2} = e^{-\frac{\Delta t}{\varepsilon^2}} g_{i,j}^n + \left(1 - e^{-\frac{\Delta t}{\varepsilon^2}}\right) \bar{\mathbb{D}}_j \rho_i^n;$

Step (i)c $\rho_i^{n+1/2} = \rho_i^n;$

Step (ii)a $f_{i,j}^{n+1} = f_{i,j}^{n+1/2} + \Delta t \mathbb{D}_j g_{i,j}^{n+1/2};$

Step (ii)b $g_{i,j}^{n+1} = g_{i,j}^{n+1/2};$

Step (iii)c by a right-rectangluar rule: $\rho_i^{n+1} = \frac{\Delta v}{2} \sum_{j=0}^{j-2} f_{i,j}^{n+1}.$

Kinetic equation

The space-derivatives $\mathbb{D}$ and $\bar{\mathbb{D}}$

In the fully-relaxed scheme, we obtain $\rho_i^{n+1} = \rho_i^n + \Delta t \mathbb{D}_j \bar{\mathbb{D}}_j \rho_i^n$. In order to recover the classical three-point centered scheme for the heat equation, needed for the scheme to be stable, $\mathbb{D}$ and $\bar{\mathbb{D}}$ must be taken in aternate direction. We define, therefore:

$$[\mathbb{D}_j \varphi]_i = \frac{1}{\Delta x} \begin{cases} -v_j \left(\varphi_i - \varphi_{i-1}\right) & \text{if } v_j \in V_+ \\ -v_j \left(\varphi_{i+1} - \varphi_i\right) & \text{if } v_j \in V_- \end{cases}$$

$$[\bar{\mathbb{D}}_j \varphi]_i = \frac{1}{\Delta x} \begin{cases} -v_j \left(\varphi_{i+1} - \varphi_i\right) & \text{if } \in V_+ \\ -v_j \left(\varphi_i - \varphi_{i-1}\right) & \text{if } v_j \in V_- \end{cases}$$

where we have also introduced $V_\pm = \{j \in \{0, \dots, N_v - 1\}$ such that $v_j \in \mathbb{R}_\pm\}$.

Boundary conditions

Boundary conditions should enforce mass conservation during the advection step:

$$\sum_{i=1}^{N_x-2} \sum_j \mathbb{D}_j g_{i,j}^{n+1/2} = 0 \Longleftarrow \begin{cases} g_{0,k}^{n+1/2} = \frac{-1}{v_k \, \#[V_+]} \sum_{v_j \in V_-} v_j g_{1,j}^{n+1/2} & \text{for } k \in V_+ \\ g_{N_x-1,k}^{n+1/2} = \frac{-1}{v_k \, \#[V_-]} \sum_{v_j \in V_+} v_j g_{N_x-2,j}^{n+1/2} & \text{for } k \in V_- . \end{cases}$$

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Outline

Numerics for the first-order closure

We recall the first-order closure (dropping ε-dependency):

$$\begin{aligned} \partial_t \rho + \partial_x J &= 0 \\ \varepsilon^2 \partial_t J + \partial_x \left[\rho \psi \left(\frac{\varepsilon J}{\rho} \right) \right] &= -J \end{aligned}$$

Strategy

We introduce a new unknown $z(t,x)$ and two new parameters λ and α; the non-linear equation for the first moment is now an advection equation and the non-linearities only appear at a right hand side:

$$\begin{pmatrix} \partial_t t & \partial_x & 0 \\ 0 & \varepsilon^2 \partial_t & \partial_x \\ 0 & \varepsilon^2 \lambda^2 \partial_x & \partial_t \end{pmatrix} \begin{pmatrix} \rho \\ J \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -J \\ \frac{1}{\alpha} \left(\rho \psi(u) - z \right) \end{pmatrix},$$

with $u = \frac{\varepsilon J}{\rho}$. As $\alpha \to 0$, this system relaxes towards the original system.

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Numerics for the first-order closure

Diagonalization

We diagonalize it by means of a linear transformation of its unknowns ($\mu = \varepsilon\lambda$)

$$\begin{pmatrix} \rho \\ J \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\mu^2} & \frac{1}{\mu^2} & \frac{1}{\mu^2} \\ 0 & \frac{1}{\varepsilon\mu} & -\frac{1}{\varepsilon\mu} \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_+ \\ f_- \end{pmatrix},$$

Splitting

then apply splitting technique between the α-relaxations and the ε-relaxations:

$$\begin{pmatrix} \partial_t & 0 & 0 \\ 0 & \partial_t + \frac{\mu}{\varepsilon}\partial_x & 0 \\ 0 & 0 & \partial_t - \frac{\mu}{\varepsilon}\partial_x \end{pmatrix} \begin{pmatrix} f_0 \\ f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha}\left(\rho\psi(u) - z\right) \\ -\frac{f_+}{\varepsilon^2} + \frac{z}{2\varepsilon^2} + \frac{1}{2\alpha}\left(\rho\psi(u) - z\right) \\ -\frac{f_-}{\varepsilon^2} + \frac{z}{2\varepsilon^2} + \frac{1}{2\alpha}\left(\rho\psi(u) - z\right) \end{pmatrix}.$$

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Numerics for the first-order closure

Stiffness in Step 1.

Step 1 is again stiff as $\varepsilon \to 0$:

$$\partial_t f_\pm \pm \frac{\mu}{\varepsilon} \partial_x f_\pm = -\frac{1}{\varepsilon^2} \left[f_\pm - \frac{z}{2} \right],$$

which means that $f_\pm$ is relaxed towards $\frac{z}{2}$, so we apply the same strategy as before and split $f_\pm$ into the following sum:

$$f_\pm = \frac{z}{2} + \varepsilon g_\pm$$

and follow the same calculations as before.

Numerics for the first-order closure

Solving **Step 1.**

Developping all the computations and rewriting the system in the original variables we get:

$$z^{n+1/2} = z^n + \frac{\varepsilon(1-e^{-\Delta t/\varepsilon^2})}{2}\left(\bar{\mathbb{D}}_+(z^n) + \bar{\mathbb{D}}_-(z^n)\right) + \Delta t\left[\mathbb{D}_+\left(e^{-\Delta t/\varepsilon^2}\frac{\mu J^n}{2} + (1-e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_+(z^n)}{2}\right) + \mathbb{D}_-\left(e^{-\Delta t/\varepsilon^2}\frac{(-\mu J^n)}{2} + (1-e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_-(z^n)}{2}\right)\right]$$

$$J^{n+1/2} = e^{-\Delta t/\varepsilon^2}J^n + \frac{1-e^{-\Delta t/\varepsilon^2}}{2\mu}\left(\bar{\mathbb{D}}_+(z^n) - \bar{\mathbb{D}}_-(z^n)\right) + \frac{\Delta t}{\varepsilon\mu}\left[\mathbb{D}_+\left(e^{-\Delta t/\varepsilon^2}\frac{\mu J^n}{2} + (1-e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_+(z^n)}{2}\right) - \mathbb{D}_-\left(e^{-\Delta t/\varepsilon^2}\frac{(-\mu J^n)}{2} + (1-e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_-(z^n)}{2}\right)\right]$$

$$\rho^{n+1/2} = \rho^n + \frac{\Delta t}{\mu^2}\left(\mathbb{D}_+\left(e^{-\Delta t/\varepsilon^2}\frac{\mu J^n}{2} + (1-e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_+(z^n)}{2}\right) + \mathbb{D}_-\left(e^{-\Delta t/\varepsilon^2}\frac{(-\mu J^n)}{2} + (1-e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_-(z^n)}{2}\right)\right).$$

Numerics for the first-order closure

Solving **Step 2.**

Step 2 just involves relaxations, and no more details are given; after reconstructing the original variables we obtain

$$
\begin{aligned}
z^{n+1} &= e^{-\Delta t/\alpha} z^{n+1/2} + (1 - e^{-\Delta t/\alpha}) \rho^{n+1/2} \psi^{n+1/2} \\
J^{n+1} &= J^{n+1/2} \\
\rho^{n+1} &= \rho^{n+1/2}.
\end{aligned}
$$

Numerics for the first-order closure

Derivatives

Discretized derivatives are subjected to upwinding and are taken in alternate directions, in order to rescue the classical three-points centered scheme for the Laplacian of the heat equation in the $(\alpha \to 0, \varepsilon \to 0)$-scheme:

$$\begin{aligned}
\left(\bar{\mathbb{D}}_+(\varphi)\right)_i &= -\frac{\mu}{\Delta x}\left(\varphi_{i+1} - \varphi_i\right) \\
\left(\mathbb{D}_+(\varphi)\right)_i &= -\frac{\mu}{\Delta x}\left(\varphi_i - \varphi_{i-1}\right) \\
\left(\bar{\mathbb{D}}_-(\varphi)\right)_i &= \frac{\mu}{\Delta x}\left(\varphi_i - \varphi_{i-1}\right) \\
\left(\mathbb{D}_-(\varphi)\right)_i &= \frac{\mu}{\Delta x}\left(\varphi_{i+1} - \varphi_i\right).
\end{aligned}$$

Boundary conditions

Homogeneous Neumann conditions are used to enforce mass conservation:

$$\rho_0^n = \rho_1^n,\ \rho_{N_x-1}^n = \rho_{N_x-2}^n,\ z_0^n = z_1^n,\ z_{N_x-1}^n = z_{N_x-2}^n,\ J_0^n = -J_1^n,\ J_{N_x-1}^n = -J_{N_x-2}^n.$$

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Numerics for the first-order closure

AP properties: the limit $\alpha \to 0$

$$
\begin{aligned}
J^{n+1} &= e^{-\Delta t/\varepsilon^2} J^n + \frac{1 - e^{-\Delta t/\varepsilon^2}}{2\mu}\left(\bar{\mathbb{D}}_+(\rho^n\psi^n) - \bar{\mathbb{D}}_-(\rho^n\psi^n)\right) \\
&\quad + \frac{\Delta t}{\varepsilon\mu}\Bigg[\mathbb{D}_+\left(e^{-\Delta t/\varepsilon^2}\frac{\mu J^n}{2} + (1 - e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_+(\rho^n\psi^n)}{2}\right) \\
&\quad - \mathbb{D}_-\left(e^{-\Delta t/\varepsilon^2}\frac{(-\mu J^n)}{2} + (1 - e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_-(\rho^n\psi^n)}{2}\right)\Bigg], \\
\rho^{n+1} &= \rho^n + \frac{\Delta t}{\mu^2}\Bigg(\mathbb{D}_+\left(e^{-\Delta t/\varepsilon^2}\frac{\mu J^n}{2} + (1 - e^{-\Delta t/\varepsilon^2})\frac{\bar{\mathbb{D}}_+(\rho^n\psi^n)}{2}\right) \\
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AP properties: the limit $\varepsilon \to 0$

$$
\rho^{n+1} = \rho^n + \psi(0)\frac{\Delta t}{(\Delta x)^2}(\rho^n_{j+1} - 2\rho^n_j + \rho^n_{j-1}).
$$

Numerics for the first-order closure

AP properties: the limit $\alpha \to 0$

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J^{n+1} &= e^{-\Delta t/\varepsilon^2} J^n + \frac{1 - e^{-\Delta t/\varepsilon^2}}{2\mu} \left(\bar{\mathbb{D}}_+(\rho^n \psi^n) - \bar{\mathbb{D}}_-(\rho^n \psi^n)\right) \\
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\rho^{n+1} = \rho^n + \psi(0) \frac{\Delta t}{(\Delta x)^2} (\rho^n_{j+1} - 2\rho^n_j + \rho^n_{j-1}).
$$

Outline

Numerics for the zeroth-order closure

We recall the the zeroth-order closure reads

$$\partial_t \rho - \partial_x \left[\frac{\rho}{\varepsilon} \mathbb{G} \left(\varepsilon \frac{\partial_x \rho}{\rho} \right) \right] = 0.$$

Strategy

The zeroth-order closure is seen as the limit $\alpha \to 0$ of the following system:

$$\begin{pmatrix} \partial_t & \partial_x \\ \frac{\mu^2}{\varepsilon^2} \partial_x & \partial_t \end{pmatrix} \begin{pmatrix} \rho \\ J \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{\alpha} \left[J + \frac{\rho}{\varepsilon} \mathbb{G} \left(\varepsilon \frac{\partial_x \rho}{\rho} \right) \right] \end{pmatrix}.$$

Diagonalization

We diagonalize the system by changing variables $f_\pm = \frac{\rho}{2} \pm \frac{\varepsilon J}{2\mu}$ thus obtaining the system

$$\partial_t f_\pm \pm \frac{\mu}{\varepsilon} \partial_x f_\pm = \frac{1}{\alpha} \left[\frac{\rho}{2} - f_\pm \mp \frac{\rho}{2\mu} \mathbb{G} \left(\varepsilon \frac{\partial_x \rho}{\rho} \right) \right].$$

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Numerics for the zeroth-order closure

Decomposition

We follow the same decomposition strategy of splitting into average and fluctuations:

$$g_\pm = \frac{1}{\varepsilon} f_\pm - \frac{1}{2\varepsilon}\rho.$$

First-order splitting strategy

We solve the resulting system $\partial_t f_\pm \pm \mu \partial_x g_\pm = \frac{1}{\alpha}\left[\frac{\rho}{2} - f_\pm \mp \frac{\rho}{2\mu}\mathbb{G}\left(\varepsilon\frac{\partial_x\rho}{\rho}\right)\right] \mp \frac{\mu}{2\varepsilon}\partial_x\rho$ by splitting it into:

Step (i) solve for a Δt-time step

$$\partial_t f_\pm = \frac{1}{\alpha}\left[\frac{\rho}{2} - f_\pm \mp \frac{\rho}{2\mu}\mathbb{G}\left(\varepsilon\frac{\partial_x\rho}{\rho}\right)\right] \mp \frac{\mu}{2\varepsilon}\partial_x\rho;$$

Step (ii) solve for a Δt-time step

$$\partial_t f_\pm \pm \frac{\mu}{\varepsilon}\partial_x f_\pm = 0, \qquad \partial_t g_\pm = 0.$$

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We follow the same decomposition strategy of splitting into average and fluctuations:

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Step (ii) solve for a Δt-time step

$$\partial_t f_{\pm} \pm \frac{\mu}{\varepsilon}\partial_x f_{\pm} = 0, \qquad \partial_t g_{\pm} = 0.$$

Numerics for the zeroth-order closure

Discretized system

Step (i)a

$$f_\pm^{n+1/2} = e^{-\Delta t/\alpha} f_\pm^n + \frac{\rho^n}{2}(1 - e^{-\Delta t/\alpha}) \left[1 \mp \frac{1}{\mu}\mathbb{G}\left(\mp\frac{\varepsilon}{\mu}\frac{\bar{\mathbb{D}}_\pm \rho^n}{\rho^n}\right)\right]$$
$$+\alpha(1 - e^{-\Delta t/\alpha})\frac{1}{2\varepsilon}\bar{\mathbb{D}}_\pm \rho^n;$$

Step (i)b

$$g_\pm^{n+1/2} = \frac{1}{\varepsilon} f_\pm^{n+1/2} - \frac{1}{2\varepsilon}\rho^{n+1/2};$$

Step (ii)a

$$f_\pm^{n+1} = f_\pm^{n+1/2} + \Delta t \mathbb{D}_\pm g_\pm^{n+1/2};$$

Step (ii)b

$$g_\pm^{n+1} = g_\pm^{n+1/2}.$$

Numerics for the zeroth-order closure

AP properties: the relaxed scheme $\alpha \to 0$

Step (i)a results into

$$f_{\pm}^{n+1/2} = \frac{\rho^n}{2}\left[1 + \frac{1}{\mu}\mathbb{G}\left(\frac{\varepsilon}{\mu}\frac{\bar{\mathbb{D}}_{\pm}\rho^n}{\rho^n}\right)\right],$$

while **Step (i)a** into

$$g_{\pm}^{n+1/2} = \frac{\rho^n}{2\mu\varepsilon}\mathbb{G}\left(\frac{\varepsilon}{\mu}\frac{\bar{\mathbb{D}}_{\pm}\rho^n}{\rho^n}\right).$$

Therefore, in terms of the mean value ρ, we have

$$\rho^{n+1} = \rho^n + \Delta t\left\{\mathbb{D}_+\left[\frac{\rho^n}{2\varepsilon\mu}\mathbb{G}\left(\frac{\varepsilon}{\mu}\frac{\bar{\mathbb{D}}_+\rho^n}{\rho^n}\right)\right] + \mathbb{D}_-\left[\frac{\rho^n}{2\varepsilon\mu}\mathbb{G}\left(\frac{\varepsilon}{\mu}\frac{\bar{\mathbb{D}}_-\rho^n}{\rho^n}\right)\right]\right\}.$$

AP properties: $\varepsilon \to 0$

As $\varepsilon \to 0$, the system relaxes to a scheme for the heat equation, as long as the derivatives $\mathbb{D}_{\pm}$ and $\bar{\mathbb{D}}_{\pm}$ are taken in alternate directions to recover the classical centered three-point scheme.

Numerics for the zeroth-order closure

AP properties: the relaxed scheme $\alpha \to 0$

Step (i)a results into

$$f_{\pm}^{n+1/2} = \frac{\rho^n}{2}\left[1 + \frac{1}{\mu}\mathbb{G}\left(\frac{\varepsilon}{\mu}\frac{\bar{\mathbb{D}}_{\pm}\rho^n}{\rho^n}\right)\right],$$

while **Step (i)a** into

$$g_{\pm}^{n+1/2} = \frac{\rho^n}{2\mu\varepsilon}\mathbb{G}\left(\frac{\varepsilon}{\mu}\frac{\bar{\mathbb{D}}_{\pm}\rho^n}{\rho^n}\right).$$

Therefore, in terms of the mean value ρ, we have

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As $\varepsilon \to 0$, the system relaxes to a scheme for the heat equation, as long as the derivatives $\mathbb{D}_{\pm}$ and $\bar{\mathbb{D}}_{\pm}$ are taken in alternate directions to recover the classical centered three-point scheme.

Outline

Kinetic solver

Relaxation to the heat equation $\varepsilon \to 0$

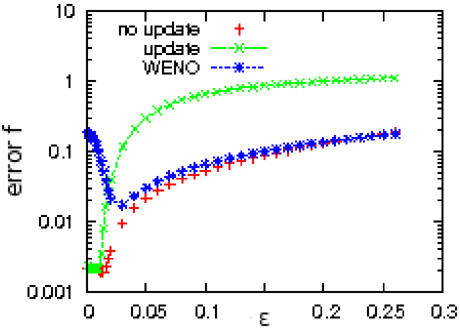

Figure: $L^2_{t,x,v}$-error of the distribution function f with respect to the solution of the heat equation with a symmetric initial datum, for a mesh of 100x100, with respect to ε.

First-order closure solver

Relaxation $\alpha \to 0$ for a fixed ε

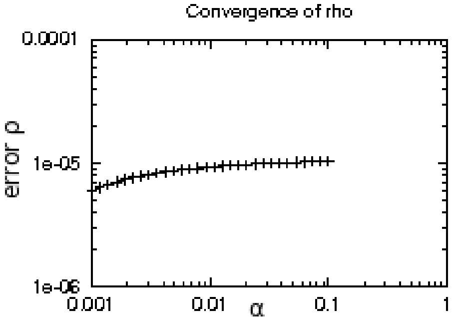

Figure: $L^2_{t,x}$-error of the densities ρ for the $\alpha > 0$ method with respect to the completely relaxed scheme $\alpha = 0$ for $\varepsilon = 0.01$.

Outline

Comparison between closures

We plot here the $L^2_{t,x,v}$-difference between the $f_\varepsilon(t,x,v)$ given by the kinetic scheme and the $\tilde{f}_\varepsilon(t,x,v)$ reconstructed from heat equation or closure schemes. As initial datum we choose a symmetric f_0 and an asymmetric f_0:

$$f_0(x,v) = \begin{cases} 2 & -0.5 \leq x \leq 0.5 \text{ and } -0.75 \leq v \leq 0.25 \quad \text{for the asymmetric i. d.} \\ 2 & -0.5 \leq x \leq 0.5 \text{ and } -0.5 \leq v \leq 0.5 \quad \text{for the symmetric i. d.} \\ 1 & \text{otherwise} \end{cases}$$

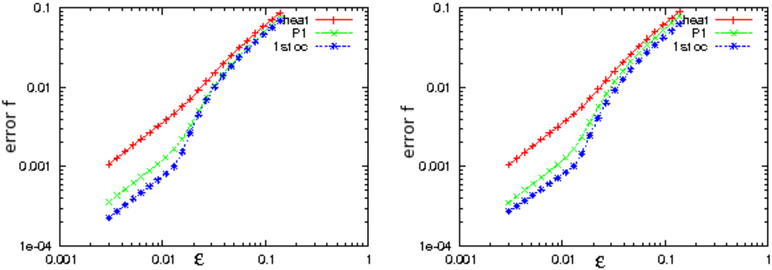

Figure: Left: symmetric initial datum. Right: asymmetric initial datum.

Outline

Su-Olson tests

The system

$$\begin{cases} \partial_t f_\varepsilon + \frac{v}{\varepsilon}\partial_x f_\varepsilon = \frac{1}{\varepsilon^2}\left(\langle f_\varepsilon\rangle - f_\varepsilon\right) + \sigma_a(\Theta - \rho) + S \\ \partial_t \Theta = \sigma_a(\rho - \Theta) \\ S = S(t,x) = \text{a given source.} \end{cases}$$

Strategy

We shall write numerical schemes for three levels, exactly as for the case of the benchmark kinetic equation:

- kinetic level;
- first-order closure;
- zeroth-order closure.

Su-Olson tests

The system

$$\begin{cases} \partial_t f_\varepsilon + \dfrac{v}{\varepsilon}\partial_x f_\varepsilon = \dfrac{1}{\varepsilon^2}\left(\langle f_\varepsilon \rangle - f_\varepsilon\right) + \sigma_a(\Theta - \rho) + S \\ \partial_t \Theta = \sigma_a(\rho - \Theta) \\ S = S(t,x) = \text{a given source.} \end{cases}$$

Strategy

We shall write numerical schemes for three levels, exactly as for the case of the benchmark kinetic equation:

- kinetic level;
- first-order closure;
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Su-Olson tests

Kinetic level

The distribution function is split into average and fluctuations $f_\varepsilon = \rho_\varepsilon + \varepsilon g_\varepsilon$, then a first-order splitting procedure is adopted:

Step (i) Solve for Δt

$$\begin{cases} g^{n+1/2} = e^{-\Delta t/\varepsilon^2} g^n - (1 - e^{-\Delta t/\varepsilon^2}) v \partial_x \rho^n, \\ f^{n+1/2} = e^{-\Delta t/\varepsilon^2} f^n + (1 - e^{-\Delta t/\varepsilon^2}) \rho^n, \\ \Theta^{n+1/2} = e^{-\sigma_a \Delta t} \Theta^n + \sigma_a (1 - e^{-\sigma_a \Delta t}) \rho^n, \\ \rho^{n+1/2} = \rho^n; \end{cases}$$

Step 2.- Solve for Δt the convection equation

$$\partial_t f + v \partial_x g = \sigma_a (\Theta - \rho) + S,$$

then update ρ^{n+1}.

Su-Olson tests

First-order closure

The system

$$\begin{cases} \partial_t \rho + \partial_x J = \sigma_a(\Theta - \rho) + S, \\ \varepsilon^2 \partial_t J + \partial_x \left[\rho \psi\left(\frac{\varepsilon J}{\rho}\right)\right] = -J \\ \partial_t \Theta = \sigma_a(\rho - \Theta). \end{cases}$$

is seen as the relaxation, as $\alpha \to 0$, of the following:

$$\begin{cases} \partial_t \rho + \partial_x J = \sigma_a(\Theta - \rho) + S, \\ \varepsilon^2 \partial_t J + \partial_x z = -J, \\ \partial_t z + \varepsilon^2 \lambda^2 \partial_x J = \dfrac{1}{\alpha}\left(\rho \psi(\varepsilon J/\rho) - z\right) \\ \partial_t \Theta = \sigma_a(\rho - \Theta). \end{cases}$$

Su-Olson tests

First-order closure: strategy

Diagonalize the system using f_0 and $f_\pm$, then split:

Step (i) Solve

$$\begin{aligned}
\partial_t f_0 &= \mu^2 \left(\sigma_a(\Theta - \rho) + S\right), \\
\partial_t f_\pm &= -\frac{f_\pm}{\varepsilon^2} + \frac{z}{2\varepsilon^2} \mp \mu \partial_x g_\pm \mp \frac{\mu}{2\varepsilon} \partial_x z, \\
\partial_t \Theta &= \sigma_a(\rho - \Theta),
\end{aligned}$$

Step (ii) Solve the ODE

$$\begin{aligned}
\partial_t f_0 &= -\frac{1}{\alpha}(\rho\psi(u) - z), \\
\partial_t f_\pm &= \frac{1}{2\alpha}(\rho\psi(u) - z), \\
\partial_t \Theta &= 0.
\end{aligned}$$

Su-Olson tests

Zeroth-order closure

$$\begin{cases} \partial_t \varrho - \partial_x\left(\frac{\varrho}{\varepsilon}\mathbb{G}\big(\varepsilon\frac{\partial_x \varrho}{\varrho}\big)\right) = \sigma_a(\Theta - \rho) + S, \\ \partial_t \Theta = \sigma_a(\rho - \Theta), \end{cases}$$

is seen as the relaxation, when α tends to 0, of

$$\begin{cases} \partial_t \rho + \partial_x J = \sigma_a(\Theta - \rho) + S, \\ \partial_t J + \frac{\mu^2}{\varepsilon^2}\partial_x \rho = -\frac{1}{\alpha}\left[J + \frac{\rho}{\varepsilon}\mathbb{G}\left(\varepsilon\frac{\partial_x \rho}{\rho}\right)\right], \\ \partial_t \Theta = \sigma_a(\rho - \Theta). \end{cases}$$

Su-Olson tests

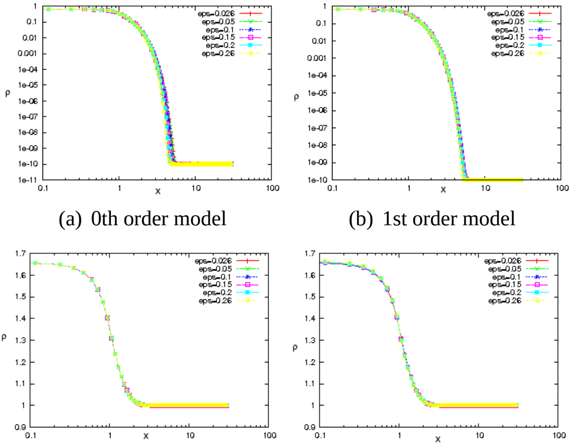
Solver

Following the same strategy as for the zeroth-order closure of the benchmark system, and relaxing the numerical scheme to $\alpha = 0$, we obtain the following

$$\begin{aligned}\Theta^{n+1} =& e^{-\sigma_a \Delta t}\Theta^n + \sigma_a(1 - e^{-\sigma_a \Delta t})\rho^n.\\ \rho^{n+1} =& \rho^n + \Delta t \left\{ \mathbb{D}_+ \left[\frac{\rho^n}{2\varepsilon\mu} \mathbb{G}\left(\frac{\varepsilon}{\mu} \frac{\bar{\mathbb{D}}_+ \rho^n}{\rho^n} \right) \right] + \mathbb{D}_- \left[\frac{\rho^n}{2\varepsilon\mu} \mathbb{G}\left(\frac{\varepsilon}{\mu} \frac{\bar{\mathbb{D}}_- \rho^n}{\rho^n} \right) \right] \right\}\\ &+ \Delta t \left(\sigma_a(\Theta^{n+1} - \rho^n) + S^n \right).\end{aligned}$$

Su-Olson tests

Numerical results



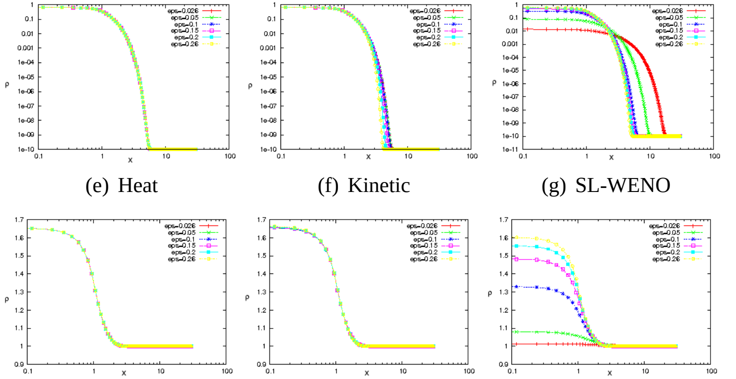
(a) 0th order model (b) 1st order model

(c) 0th order model (d) 1st order model

Figure: $t = 1$; (a) and (b): $f_0 = \rho_0 = \Theta_0 = 10^{-10}$; (c) and (d):$f_0 = \rho_0 = \Theta_0 = 1$.

Su-Olson tests

Numerical results



(e) Heat (f) Kinetic (g) SL-WENO

(h) Heat (i) Kinetic (j) SL-WENO

Figure: Su-Olson test: Comparison of the density ρ computed by the different models as ε varies at time $t = 1$ (continued).

Su-Olson tests

Numerical results

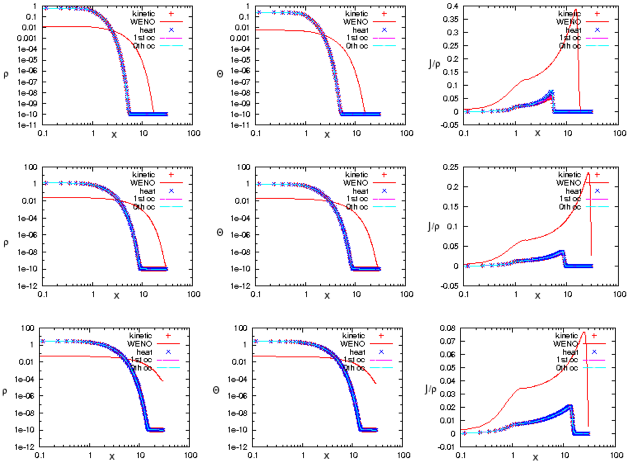


Figure: $\varepsilon = 0.026$, the initial datum is $f_0 = \rho_0 = \Theta_0 = 10^{-10}$.

Su-Olson tests

Numerical results

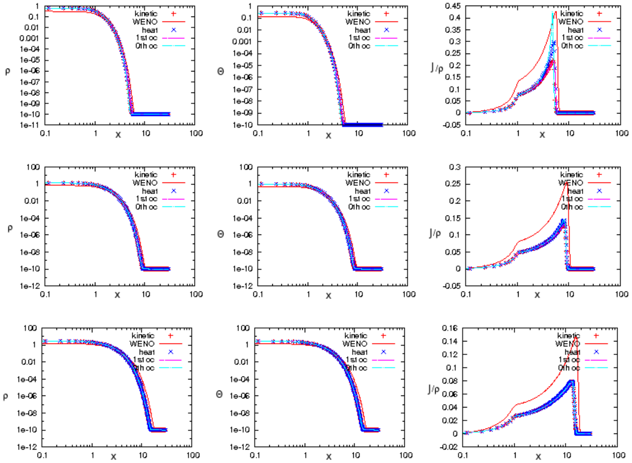


Figure: $\varepsilon = 0.1$, the initial datum is $f_0 = \rho_0 = \Theta_0 = 10^{-10}$.

Su-Olson tests

Numerical results

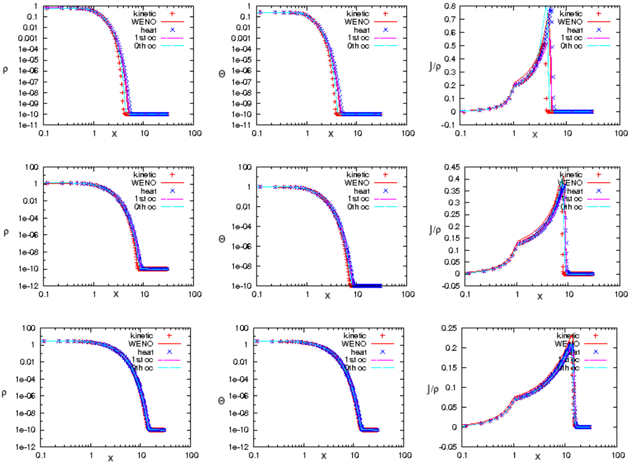


Figure: $\varepsilon = 0.26$, the initial datum is $f_0 = \rho_0 = \Theta_0 = 10^{-10}$.

Su-Olson tests

Numerical results

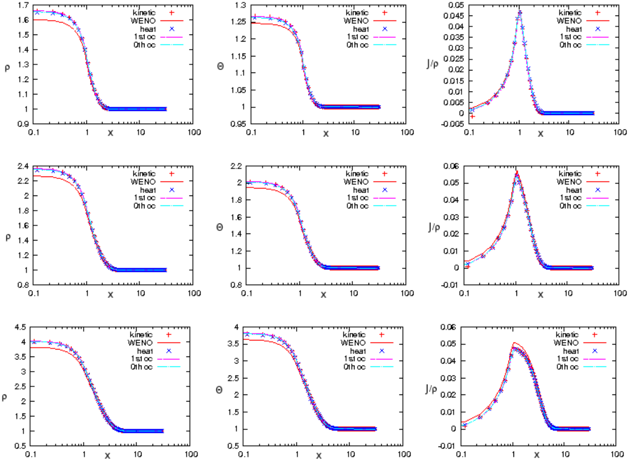


Figure: $\varepsilon = 0.26$, the initial datum is $f_0 = \rho_0 = \Theta_0 = 1$.

Su-Olson tests

Numerical results

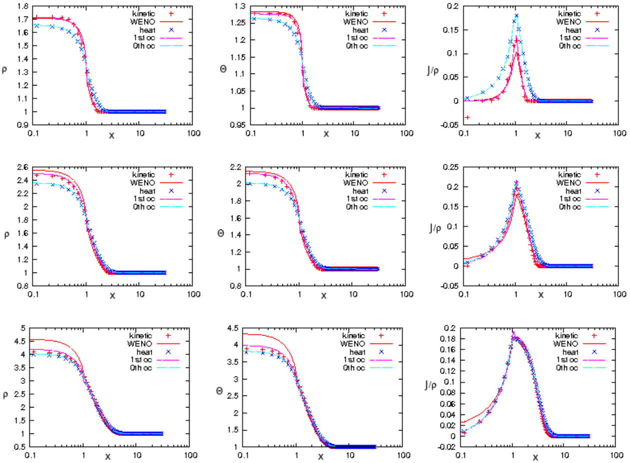


Figure: $\varepsilon = 1$, the initial data is $f_0 = \rho_0 = \Theta_0 = 1$.