A semi-Lagrangian AMR scheme for 2D transport problems in conservation form

Pep Mulet Mestre, Francesco Vecil

Universitat de València

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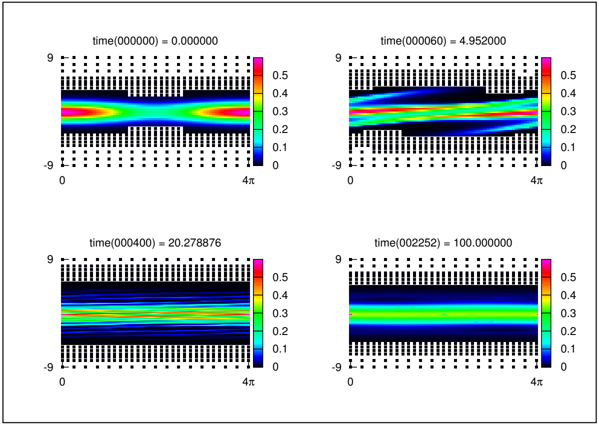
Outline

Motivation



No need for fine meshing everywhere in the domain.

$\Downarrow$

Refine only where the important information is.

Framework

Equations

In dimension N, transport equations written in conservtion form:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}\left[a_1(t, x_1, x_2)u\right] + \frac{\partial}{\partial x_2}\left[a_2(t, x_1, x_2)u\right] = 0, \qquad u(0, x_1, x_2) = u^0(x_1, x_2),$$

where $\boldsymbol{a} : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}^2$ is the advection field.

Example

The three-dimensional Vlasov-Maxwell equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v}(\boldsymbol{p}) \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0, \quad \boldsymbol{v}(\boldsymbol{p}) := \frac{\boldsymbol{p}}{m\sqrt{1 + \frac{|\boldsymbol{p}|^2}{m^2 c^2}}}, \quad \boldsymbol{F} := -e(\boldsymbol{E} + \boldsymbol{v}(\boldsymbol{p}) \wedge \boldsymbol{B}),$$

describes the evolution of $f(t, \boldsymbol{x}, \boldsymbol{p})$, typically representing the concentration of electrons or holes at position $\boldsymbol{x}$ and momentum $\boldsymbol{p}$.

Features

Shocks, large gradients, filamentation, microscopic structures.

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Grid hierarchy

Resolution levels

We define $L+1$ resolution levels: the coarsest is $\ell = 0$, the finest $\ell = L$. In 1D, the meshes are

$$x_{\ell,j} = x_{\min} + j\Delta x_\ell, \qquad \Delta x_\ell = \frac{x_{\max} - x_{\min}}{2^\ell N_0}.$$

Grid

The ℓ-grid at time t^n is

$$G_\ell^n = \{x_{\ell,j}\}_{j \in \mathcal{G}_\ell^n}.$$

We are interested in

$$\mathcal{G}_\ell^n \subseteq \prod_{i=1}^{N} \{0, \dots, N_{i,\ell}\}.$$

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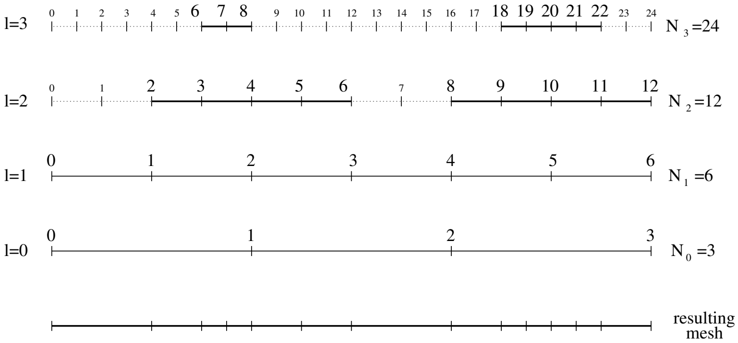
Nesting condition

We are interested in *nested* meshes:



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Ghost points are added outside the ℓ-grids to take into account the boundary conditions for the time integration.

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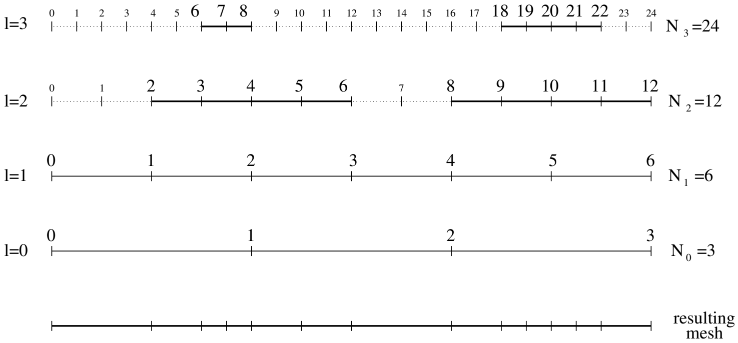

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Grid selection

The grid selection is based on two criteria: *prediction* and *gradient*.

The *prediction* criterion

Let $\{u_{\ell,j}\}_{j\in\mathcal{G}_\ell^n}$ the point values at reslution level ℓ. Let $p[u_\ell]$ an interpolator at resolution level ℓ. We keep point $x_{\ell,j}$ for refinement if

$$|p[u_{\ell-1}](x_{\ell,j}) - u_{\ell,j}| > \tau_p.$$

The *gradient* criterion

We estimate the gradient at point $x_{\ell,j}$ by means of the discrete gradient computed at resolution level $\ell-1$: if it is over a certain tolerance parameter $\tau_{d,\ell-1}$, then point $x_{\ell,j}$ is selected for refinement.

Reconstruction

Once the grid has been selected, reconstruct by means of an interpolator the point values that are not assigned yet.

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1D semi-Lagrangian strategy

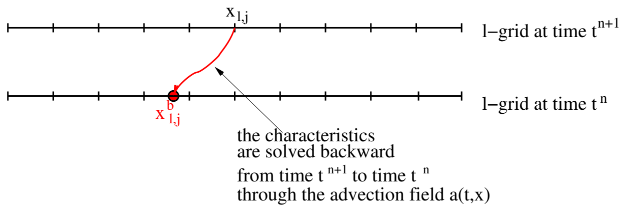
Characteristic-based solution

The solution to the PDE $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[a(t,x)u\right] = 0, \qquad u(0,x) = u^0(x)$$

is given by $$u(t,x) = u\left(s, \mathcal{X}(s;t,x)\right) J(s;t,x),$$

with $\mathcal{X}(s;t,x)$ the characteristic at time s, starting from x at time t:

$$\frac{\mathrm{d}\mathcal{X}(s;t,x)}{\mathrm{d}s} = a\left(s, \mathcal{X}(s;t,x)\right), \qquad \mathcal{X}(t;t,x) = x, \qquad J(s;t,x) := \frac{\partial \mathcal{X}(s;t,x)}{\partial x}.$$



1D semi-Lagrangian strategy

What do we need?

- A solver for the characteristics $\mathcal{X}(s;t,x)$: **Runge-Kutta**.
- An approximation for the Jacobian through **cenetred finite differences**:

$$J(s;t,x)=\frac{\partial\mathcal{X}(s;t,x)}{\partial x}\approx\frac{\tilde{\mathcal{X}}(s;t,x+\delta x)-\tilde{\mathcal{X}}(s;t,x-\delta x)}{2\delta x},\quad \delta x=10^{-m}\Delta x_\ell.$$

- An interpolator to reconstruct $u(s,\mathcal{X}(s;t,x))$: **PVWENO** (Point-Value Weighted Essentially Non-Oscillatory).

Error estimate

The local truncation error can be estimated

$$E=\underbrace{\underbrace{\mathcal{O}\left(\Delta t^{s+1}\right)}_{\text{Runge-Kutta}}+\underbrace{\mathcal{O}\left(\Delta x_\ell^2\right)}_{\text{Jacobian}}}_{\text{approximation of the characteristics}}+\underbrace{\mathcal{O}\left(\Delta x_\ell^{2r}\right)}_{\text{PVWENO (advection)}}.$$

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The 2D case

Grid hierarchy and selection

We do not give details, but we apply strategies similar to the 1D case.

The 2D PDE

We solve the 2D PDE $\quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}(a_1 u) + \frac{\partial}{\partial x_2}(a_2 u) = 0$

by splitting the (x_1, x_2)-domain thanks to the second-order Strang scheme:

- Solve for a $\frac{\Delta t}{2}$ time step $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}(a_1 u) = 0$;
- Solve for a Δt time step $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_2}(a_2 u) = 0$;
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Error estimate

The Strang splitting constrains the accuracy:

$$E = \mathcal{O}\left(\Delta t^{\min(s+1,3)}\right) + \underbrace{\mathcal{O}(\Delta x_{1,\ell}^2) + \mathcal{O}(\Delta x_{2,\ell}^2)}_{\text{approximated Jacobian}} + \underbrace{\mathcal{O}(\Delta x_{1,\ell}^{2r}) + \mathcal{O}(\Delta x_{2,\ell}^{2r})}_{\text{PVWENO}}.$$

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Outline

Goals

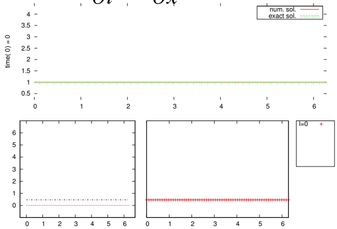
The Adaptive-Mesh-Refinement (AMR) framework is compared to the equivalent Fixed-Mesh (FM) results.

Of course, AMR cannot be more accurate than FM. Rather, it achieves faster computational times in exchange of a loss of precision.

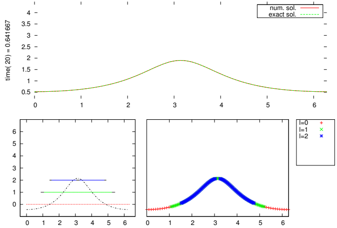
Outline

Variable-coefficient advection

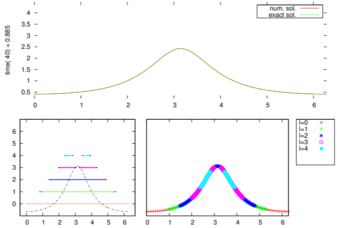
The test case $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(\sin(x)\,u) = 0$ produces a blow-up.



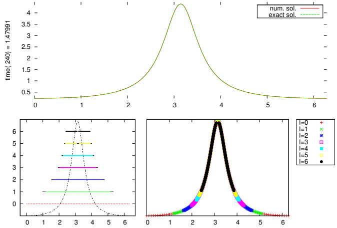
(a) at time 0



(b) at time 0.65



(c) at time 0.89



(d) at time 1.48

Variable-coefficient advection

Speedup

For parameters

$$
\begin{aligned}
N_0 &= 128 && \text{points at } \ell = 0 \\
L &= 4 && \text{number of resolution levels} \\
\Delta t_0 &= 0.125 && \text{maximum time step} \\
\tau_p &= 10^{-4} && \textit{prediction}\text{-criterion tolerance} \\
\tau_{d,0} &= 0.5 && \textit{gradient}\text{-criterion tolerance,}
\end{aligned}
$$

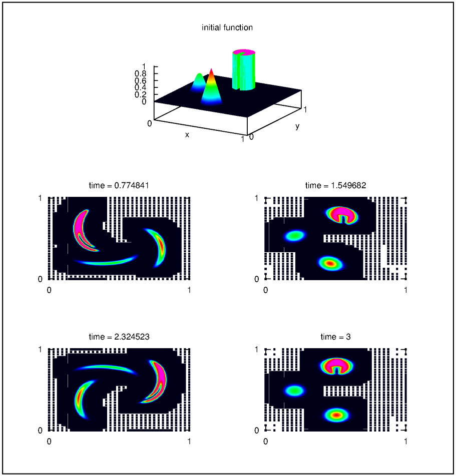
AMR reaches a speedup of 35 times with respect to the equivalently-resolved FM, with a loss of precision from 10^{-9} to roughly 10^{-6} (the L^2-error w.r.t. the analytical solution).

Outline

Deformation flows

The system

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x}\left[\sin^2(\pi x)\sin(2\pi y)g(t)f\right]+\frac{\partial}{\partial y}\left[-\sin^2(\pi y)\sin(2\pi x)g(t)f\right]=0,\quad (x,y)\in[0,1]^2,$$



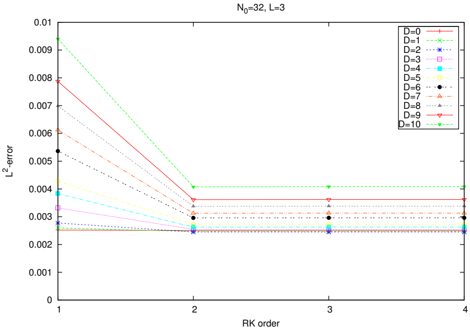
for $g(t)=\cos\left(\frac{\pi t}{T}\right)$, periodically recovers the initial datum after alternate clockwise and counterclockwise twistings.

Performances

Speedup ≈ 2.

Deformation flows

The ODE integrator for the characteristics



As announced by the error estimate

$$E = \mathcal{O}(\Delta x_{1,\ell}^2) + \mathcal{O}(\Delta x_{2,\ell}^2) + \mathcal{O}\left(\Delta t^{\min(s+1,3)}\right)$$

the Strang-splitting order constrains the accuracy.

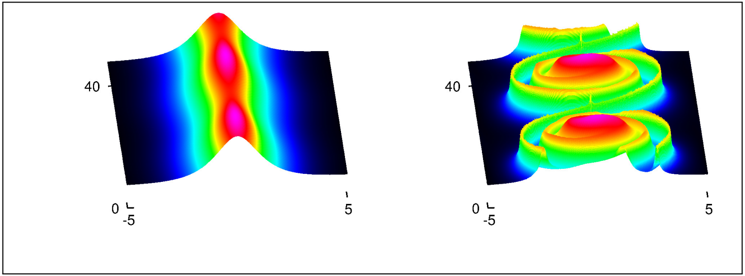
Kelvin-Hemlholtz instabilities

The model

The *guiding-center* model (omitting some details)

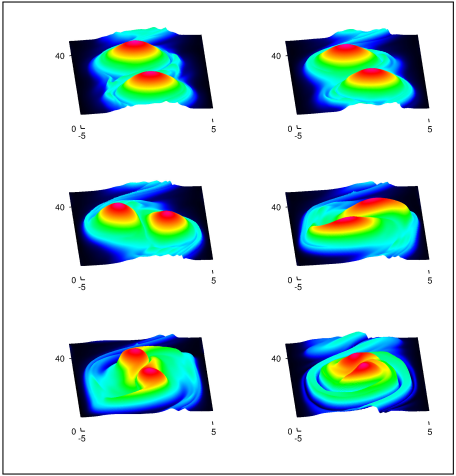
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1}\left[\frac{\partial \Phi}{\partial x_2}\rho\right] + \frac{\partial}{\partial x_2}\left[-\frac{\partial \Phi}{\partial x_1}\rho\right] = 0, \qquad \Delta_{x_1,x_2}\Phi = \rho,$$

for initial condition $\rho(0,x_1,x_2) = 1.5 \operatorname{sech}\left(\frac{x_2}{0.9}\right) \cdot (1 + 0.08 \sin(2kx_1))$,
periodic x_1- and Dirichlet x_2-boundaries, produces vortices and filamentation.

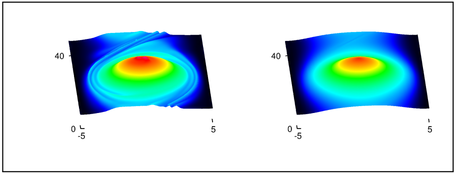


(f) first instability

Kelvin-Hemlholtz instabilities



(g) second instability



(h) steady state

SPEEDUP

The AMR strategy achieves a speedup of about 2.8, for parameters $(N_{\star,0}, L) = (32, 4)$ (the maximum resolution is 512×512), $\tau_p = 10^{-3}$, $\tau_{d,0} = 1.5$.

GRÀCIES!
HVALA!
DANKE!
ĎAKUJEM!
DĚKUJI!